7.1 Properties of z-Transform

Let

\[ x_1[n] \xrightarrow{z} X_1(z) \quad \text{ROC} = R_{x_1} \]
\[ x_2[n] \xrightarrow{z} X_2(z) \quad \text{ROC} = R_{x_2} \]

**Linearity**

\[ ax_1[n] + bx_2[n] \xrightarrow{z} aX_1(z) + bX_2(z) \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]

which follows from definition of z-transform.

**Ex:**

\[ x[n] = a^n u[n] - a^n u[n - N] \quad (N \geq 0) \]

Let

\[ x_1[n] = a^n u[n] \xrightarrow{z} X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC} = |z| > |a| \]
\[ x_2[n] = a^n u[n - N] \xrightarrow{z} X_2(z) = \frac{a^N z^{-N}}{1 - az^{-1}} \quad \text{ROC} = |z| > |a| \]

Then

\[ x[n] = x_1[n] - x_2[n] \xrightarrow{z} X(z) = \frac{1 - a^N z^{-N}}{1 - az^{-1}} \]

However, the pole at \( z = a \) of \( X(z) \) is cancelled out by the zero at \( z = a \). So the ROC of \( X(z) \) is the entire \( z \)-plane when \( N = 0, 1 \). The ROC is \( |z| > 0 \) when \( N > 1 \).

**Time Shifting**

\[ x[n - n_0] \xrightarrow{z} z^{-n_0} X(z) \]

Let \( y[n] = x[n - n_0] \),

\[ Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} = \sum_{k=-m}^{\infty} x[k] z^{-k-n_0} = z^{-n_0} X(z) \]

The ROC of \( Y(z) \) is the same as \( X(z) \) except that there are possible pole additions or deletions at \( z = 0 \) or \( z = \infty \).
Multiplication by an Exponential Sequence

Let \( y[n] = z_0^n x[n] \),

\[
Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left( \frac{z}{z_0} \right)^{-n} = X \left( \frac{z}{z_0} \right)
\]

The consequence is pole and zero locations are scaled by \( z_0 \). If the ROC of \( X(z) \) is \( r_R < |z| < r_L \), then the ROC of \( Y(z) \) is

\[
r_R < \left| \frac{z}{z_0} \right| < r_L \text{, i.e., } |z_0| r_R < |z| < |z_0| r_L
\]

Differentiation of \( X(z) \)

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}
\]

\[
-\frac{dX(z)}{dz} = -z \sum_{n=-\infty}^{\infty} (n) x[n] z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n] z^{-n}
\]

So

\[
x[n] \overset{z}{\leftrightarrow} -\frac{dX(z)}{dz} \quad \text{ROC} = R_x\]

Conjugation of a Complex Sequence

\[
x^*[n] \overset{z}{\leftrightarrow} X^*(z^*) \quad \text{ROC} = R_x
\]

Let \( y[n] = x^*[n] \), then

\[
Y(z) = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = \left( \sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right)^* = X(z^*)^*
\]

Time Reversal

\[
x^*[-n] \overset{z}{\leftrightarrow} X^*(1/z^*)
\]

Let \( y[n] = x^*[-n] \), then

\[
Y(z) = \sum_{n=-\infty}^{\infty} x^*[-n] z^{-n} = \left( \sum_{n=-\infty}^{\infty} x[-n] (z^*)^{-n} \right)^* = \left( \sum_{k=-\infty}^{\infty} x[k] (1/z^*)^{-k} \right)^* = X^*(1/z^*)
\]

If the ROC of \( X(z) \) is \( r_R < |z| < r_L \), then the ROC of \( Y(z) \) is

\[
r_R < |1/z^*| < r_L \text{, i.e., } \frac{1}{r_L} < |z| < \frac{1}{r_R}
\]

When the time reversal is without conjugation, it is easy to show

\[
x[-n] \overset{z}{\leftrightarrow} X(1/z) \quad \frac{1}{r_L} < |z| < \frac{1}{r_R}
\]
Convolution of Sequences

Let

\[ y[n] = x_1[n] \ast x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \]

then

\[ Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n} \]

\[ = \sum_{k=-\infty}^{\infty} \left( x_1[k] z^{-k} \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-(n-k)} \right) \]

\[ = \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} X_2(z) \]

\[ = X_1(z) X_2(z) \]

The ROC of \( Y(z) \) contains \( R_{x_1} \cap R_{x_2} \), because cancellation of zeros and poles may result in an ROC larger than \( R_{x_1} \cap R_{x_2} \).

Initial Value Theorem

If \( x[n] = 0 \) for \( n < 0 \),

\[ x[0] = \lim_{z \to \infty} X(z) \] (7.1)

Proof.

\[ \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = x[0] + \sum_{n=1}^{\infty} x[n] \lim_{z \to \infty} z^{-n} \]

\[ = x[0] \]

7.2 Sampling

Sampling implements the representation of a continuous-time signal by a discrete-time signal, which enables the processing of signal by digital computers.

Q: given \( x(t) \), need we have \( x(t) \) at all times in order to represent \( x(t) \) at all times? A: No.

Q: might \( x(t) \) be redundant, so we can store only a portion of \( x(t) \) without losing any information? A: Yes.

Q: might this be useful when using computers? A: Yes.

Periodic Sampling

An ideal continuous-to-discrete-time (C/D) converter using periodic sampling is shown in Fig. 7.1. It can be mathematically described as

\[ x[n] = x_c(nT) \quad -\infty < n < \infty \] (7.2)

where \( x_c(t) \) is the input continuous time signal and \( x[n] \) is the output discrete time signal. \( T \) is the sampling period. \( f_s = \frac{1}{T} \) is the sampling frequency in samples/sec. \( \Omega_s = \frac{2\pi}{T} \) is the sampling frequency in radians/sec.

Question: Can we recover \( x_c(t) \) from \( x[n] \)?
Answer: It depends. If $x_c(t)$ has certain characteristics and $T$ is small enough then yes. Otherwise, no.

A mathematically convenient representation of the ideal C/D converter.

Fig. 7.2 shows a mathematically convenient representation of the ideal C/D converter. In the diagram,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = x_c(t) s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)$$

Frequency Domain Representation of Sampling

We now study the frequency domain representations of the signals and sequence in Fig. 7.2.

Recall we use variable $\omega$ for discrete-time (normalized) radian frequency. We use variable $\Omega$ for continuous-time radian frequency.

Let

$$x_c(t) \leftrightarrow X_c(j\Omega)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$x_s(t) \leftrightarrow X_s(j\Omega)$$

$$x[n] \leftrightarrow X(e^{j\omega})$$
then we get

\[
X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{2\pi} \sum_{k=\infty}^{\infty} \delta(j(\Omega-k\Omega_s)) \cdot \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(t-nT) \rightarrow \frac{2\pi}{T} \sum_{k=\infty}^{\infty} \delta(j(\Omega-k\Omega_s))
\]

\[\text{(7.3)}\]

\[
= \frac{1}{T} \sum_{k=\infty}^{\infty} X_c(j(\Omega-k\Omega_s))
\]

\[
= \frac{1}{T} \sum_{k=\infty}^{\infty} X_c(j(\Omega-k\Omega_s))
\]

\[\text{(7.3)}\]

\[X_s(j\Omega)\) can be interpreted as a superimposed version of periodically repeated \(X_c(j\Omega)\). An example is shown in Fig. 7.3. Notice there might be overlapping among \(X_c(j(\Omega-k\Omega_s))\) \((-\infty < k < \infty\)). When overlapping happens, aliasing occurs

\[\text{Figure 7.3: Frequency domain representation of sampling.}\]

in \(X_s(j\Omega)\), with respect to \(X_c(j\Omega)\). If \(X_c(j\Omega) = 0\) for \(|\Omega| \geq \Omega_N\), from Fig. 7.3, aliasing can be avoided if \(\Omega_s - \Omega_N > \Omega_N\).

In the non-aliasing case (\(\Omega_s > 2\Omega_N\)), we can design an ideal lowpass filter

\[
H_r(j\Omega) = \begin{cases} T & |\Omega| < \Omega_c \\ 0 & |\Omega| \geq \Omega_c \end{cases}
\]

\[\text{(7.4)}\]

where \(\Omega_c\) is the cutoff frequency of the lowpass filter. By applying the lowpass filter on the non-aliased \(x_s(t)\) and use an appropriate cutoff frequency \(\Omega_c\) (usually \(\Omega_c = \frac{\Omega_s}{2}\)), we can reconstruct \(x_c(t)\). Two conditions for reconstruction must be satisfied:

1. \(\Omega_N\) (the highest frequency of the signal) exists.
2. \(\Omega_s > 2\Omega_N\)
In the aliasing case ($\Omega_s < 2\Omega_N$), $X_c(j\Omega)$ is no longer recoverable by LPF because of overlapping of frequency components. This is a form of non-linear distortion shown as aliasing. Some frequency components are aliases of those before sampling.

By certain types of filter, we can also get $X'_c(j\Omega)$ such that

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X'_c(j(\Omega - k\Omega_s))$$

However, is this solution uniquely determined? Apparently, $X'_c(j\Omega) = H_r(j\Omega)X_s(j\Omega)$ is always a solution for the above equation and $X_c(j\Omega)$ is another solution. But $X'_c(j\Omega) \neq X_c(j\Omega)$ if there is aliasing.

Nyquist Sampling Theorem states that only when there is no aliasing, can $x_c(t)$ be uniquely determined.

**Theorem 7.1 (Nyquist Sampling Theorem).** Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT) \quad n = 0, \pm 1, \pm 2, \cdots$$

if

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N \quad (7.5)$$

where $\Omega_N$ is called the Nyquist frequency and $2\Omega_N$ is called the Nyquist rate, which must be exceeded by $\Omega_s$ for ideal reconstruction of $x_c(t)$.

**Fourier transform of $x[n]$.**

Since

$$X_s(j\Omega) = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \right\}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\mathcal{F}\{\delta(t - nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)e^{-jn\Omega T}$$

and

$$x[n] = x_c(nT)$$

we get

$$X_s(j\Omega) = X(e^{j\omega}) \big|_{\omega = \Omega T}$$

So

$$X(e^{j\omega}) = X_s(j\Omega) \big|_{\Omega = \frac{\omega}{T}} \quad (7.6)$$

which is a frequency axis scaled version of $X_s(j\Omega)$ by $\frac{1}{T}$, the sampling frequency.

**Ex:** Sampling a sinusoidal signal with aliasing.

$$x_c(t) = \cos(4000\pi t), \quad T = \frac{1}{1500}$$

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

Then

$$x[n] \propto \cos\left(\frac{2\pi}{3}n\right)$$

Fig. 7.4 shows the frequency domain representations during the sampling process.

---

1 We can design certain filter to produce $X_c(j\Omega)$, even though it may not be as trivial as the lowpass filter.
Figure 7.4: Sampling a sinusoidal signal with aliasing.