Problem # 1 (20 points)

Consider a system $H(z)$ with input $x[n]$ and output $y[n]$ that is characterized by the following input/output relationship in the time domain:

$$y[n] = ay[n - 1] + x[n]$$

where we are given that $|a| < 1$. As can be seen, the value of $y$ at time $n$ depends on the value of $y$ at the previous time $n - 1$.

In some applications, a computation of $y[n]$ from $x[n]$ is needed where the value of $y$ at a given time $n$ depends on values of $y$ no later than at some time before the previous time step $n - 1$. For example, the value of $y[n]$ might depend on $y[n - 2]$ and earlier samples of $y$, but not on $y[n - 1]$. In this problem, you will design and analyze such a system.

(1a) (4 points) Come up with a new system $H_p(z)$ that has exactly the same impulse and frequency response as $H(z)$, but the value of $y$ at time $n$ depends on values of $y$ at times no later than $n - 4$. Express your system both as a time domain relation (3 points) and in the frequency domain (1 point).

$$y[n] = ay[n - 1] + x[n]$$

$$= a(ay[n - 2] + x[n - 1]) + x[n]$$

$$= a(a(ay[n - 3] + x[n - 2]) + x[n - 1]) + x[n]$$

$$= a(a(a(ay[n - 4] + x[n - 3]) + x[n - 2]) + x[n - 1]) + x[n]$$

$$= a^4y[n - 4] + a^3x[n - 3] + a^2x[n - 2] + ax[n - 1] + x[n]$$

Therefore, the resulting system is

$$H_p(z) = \frac{Y(z)}{X(z)} = \frac{1 + az^{-1} + a^2z^{-2} + a^3z^{-3}}{1 - a^4z^{-4}}$$

(1b) (4 points) Going from $H(z)$ to $H_p(z)$ can be seen as an application of a third system $G(z) = H_p(z)/H(z)$. Come up with an expression for $G(z)$, where it can be seen how it creates the $H_p(z)$ producing a $y$ with the longer temporal dependence.
Note that we can also represent $H_p(z)$ as follows:

$$H_p(z) = \frac{Y(z)}{X(z)} = \frac{(1 + az^{-1})(1+a^2z^{-2})}{(1-a^2z^{-2})(1+a^2z^{-2})}$$

and since

$$H(z) = \frac{1}{1-az^{-1}}$$

then

$$G(z) = H_p(z)/H(z) = \frac{(1-a^2z^{-2})(1+a^2z^{-2})}{(1-a^2z^{-2})(1+a^2z^{-2})} = 1$$

(1c) (4 points) Give the magnitude (1 point) and phase response (1 point) of $G(z)$ and say if $G(z)$ is LTI (2 points).

The magnitude is $|H(z)| = 1$

and the phase is $\angle H(z) = 0$

Since $G(z)$ is just a constant unity, yes it is LTI.

(1d) (4 points) Given your answers to part c, explain in the $z$-domain what precisely $G(z)$ is doing (2 points) and how it is producing the dependence delaying effect above (2 points)?

We are multiplying $H(z)$ by terms such as $(1+az^{-1})/(1+az^{-1})$ or $(1+a^2z^{-2})/(1+a^2z^{-2})$ which have themselves a pole-zero cancellation. This therefore does not affect the magnitude or phase of the system, but when converting back to the time domain produces a recursion for $y$ with a longer temporal dependence.

(1e) (4 points) Would a system that is implemented as the cascade of $G(z)$ with $H(z)$ have the same effect? Explain and justify your answer.

No. In a cascade system, one would first apply $H(z)$ and then $G(z)$. In such a cascade, the first system $H(z)$ would still have the short temporal dependence.

Problem # 2 (20 points)

You’ve seen in class that we defined a generalized linear phase system $H(e^{j\omega})$ as one that is defined as follows:

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha \omega + j\beta}$$

where $A(e^{j\omega})$ is strictly real periodic function of $\omega$.

We also know that all such $H(e^{j\omega})$ are periodic with period $2\pi$. Must it be the case that $A(e^{j\omega})$ is also periodic with period $2\pi$? If so, prove it. If not, come up with an $H(e^{j\omega})$ where $A(e^{j\omega})$ is not strictly periodic with period $2\pi$? (Note: if it is periodic with period $\pi$ then it is clearly periodic with period $2\pi$ so we are not talking about this case. We are also not talking about the case when $A(e^{j\omega})$ is a constant.).

Consider 

$$H(z) = 1 + z^{-1}$$

Then 

$$H(e^{j\omega}) = e^{-j\omega/2}(2\cos(\omega/2))$$

so that $A$ is periodic but with period $4\pi$.

Problem # 3 (30 points)

Again, consider the definition of generalized linear phase system $H(e^{j\omega})$:

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha \omega + j\beta}$$
In this problem, we assume that $\alpha = M/2$ is either an integer or an integer plus 0.5 (i.e., $M$ is an integer) and (for simplicity) that $\beta = 0$.

(3a) (10 points) Prove that if the conditions above hold, then the resulting $h[n]$ must be symmetric about $\alpha$, i.e.,

$$h[2\alpha - n] = h[n]$$

In your prove, show the equations leading to your solution.

Let $h[n]$ be the inverse Fourier transform of $H(e^{j\omega})$ and $a[n]$ be the inverse transform of $A(e^{j\omega})$. We need only show that $a[n]$ is symmetric about $n = 0$ since multiplication by $e^{-j\omega}$ in the frequency domain will delay a symmetric at $n = 0$ signal to be symmetric at $n = 2\alpha$. Taking the inverse Fourier transform, we get:

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega}) \cos(\omega n) d\omega$$

But this is clearly symmetric about $n = 0$ since the $\cos$ function is even.

(3b) (20 points) Let $H(z) = 1/(\sum_{k=0}^{M} a_k z^{-k})$ represent a causal filter with linear phase (meaning it can be represented as above). Assume that $a_0 = 1$. Prove that $H(z)$ is unstable unless $a_k = 0$ for all $k > 0$. (Hint: Think of the poles of $H(z)$, and what must be true of them in this case).

First, represent $H$ as follows.

$$H(z) = \prod_{k=1}^{n} \frac{1}{1 - d_k \ z^{-1}}.$$

Along the unit circle, it becomes:

$$H(e^{j\omega}) = \frac{1}{\prod_{k=1}^{n} (1 - d_k e^{-j\omega})}$$

We multiply $H$ by 1 in the form of

$$1 = \left(\frac{e^{j\omega/2}}{e^{j\omega/2}}\right)^M$$

leading to:

$$H(e^{j\omega}) = \frac{e^{j\omega M/2}}{\prod_{k=1}^{n} (e^{j\omega/2} - d_k e^{-j\omega/2})}$$

and in the limit as each of the $d_k$ coefficients go to 1, this becomes:

$$H(e^{j\omega}) = \frac{e^{j\omega M/2}}{\prod_{k=1}^{n} (2 \cos(\omega/2))}$$

which has the linear phase form. But in turning it into this form, we have moved the poles of $H$ to the unit circle making it unstable. Note that if $a_k = 0$ for all $k > 0$ then we get that $h[n] = \delta[n]$ which is clearly both linear phase and FIR.

Problem # 4 (15 points)
Suppose that we wish to compute the 8-point DFT of a sequence $x[n]$. Assume that we have two 4-point DFT modules available (as indicated below). Show two ways of computing the 8-point FFT using the modules available, along with a small amount of additional computation. In the first method (7.5 points), computation after the module outputs is not permitted. In the second method (7.5 points), computation before the module inputs is not permitted.

Express your answers by completing the unfinished diagram provided on the answer sheet below. Make sure to label
all inputs and outputs of each system.

Problem # 5  (15 points) An engineer designs a discrete-time low-pass filter with cut-off frequency $\omega_c = \pi/2$ of the length-$N$ signal (assume $N$ is even) $x[n]$ via the following steps.

1. Take the size-$N$ in-place decimation-in-time FFT of $x[n]$ resulting in $X[k]$, for $k = 0, 1, 2, \ldots, N - 1$.

2. Set $X[k] = 0$ for $k = N/2, N/2+1, \ldots, N - 1$ and call the resulting sequence $Y[k]$.

3. Take the corresponding inverse FFT of size $N$ of $Y[k]$ and call the resulting sequence $y[n]$.

Would this be something useful to use as a LPF? If so, give at least three reasons why it is good. If it is not a good low-pass filter, then give at least three reasons why you would not want to use it (5 points per correct reason).
This is **not** a good LPF for the following three reasons.

1. If we take the FFT of the signal, the output might be in bit-reversed order and if so, truncating one-half of the samples to zero might not be removing the high frequencies.

2. Even if the output of the FFT is in normal bit order, we are setting to zero the samples \( k = N/2, N/2 + 1, \ldots, N - 1 \). Recall, that the sample \( k = N/2 \) corresponds to \( \omega = \pi \) so we are really doing is setting to zero the negative frequency components. Therefore, this won’t be a low-pass filter at all.

3. Taking the FFT of a sequence of length \( N \) is not guaranteed to be fast given just that \( N \) is even. We would need for \( N = 2^\ell \) where \( \ell \) is an integer.

4. Note also that even if all of the above were satisfied (meaning there was no bit-reversal at the output, setting samples to zero from, say, \( \pi/2 \) to \( \pi \), and using a power of two length \( N \), such a low-pass filter might not be good because this corresponds in the time domain to circularly convolving with a standard sync function (the inverse FFT of the ideal rectangular filter). If the lengths of the signals relative to the FFT length are not set correctly, this could lead to unwanted time aliasing.

**Problem # 6** *(0 points) (but see below)*

Consider the following "proof" that \( 1 = 0 \).

\[
1 = \sqrt{1} \\
= \sqrt{(-1)(-1)} \\
= \sqrt{(-1)} \sqrt{(-1)} \\
= j \times j \\
= -1
\]

Adding 1 to both sides and dividing by 2 yields \( 1 = 0 \). Therefore, this exam is worth how many points?

What would Donald Samuel Peterson say about this?