Problem # 1 (15 points)

Consider the following three methods for “integrating” a discrete-time signal as indicated below, where \( x[n] \) is the input and \( y[n] \) is the output:

1. Running sum:

   \[
   y[n] = x[n] + y[n-1]
   \]

2. Trapezoid rule:

   \[
   y[n] = \frac{1}{2} x[n] + \frac{1}{2} x[n-1] + y[n-1]
   \]

3. Simpson’s rule:

   \[
   y[n] = \frac{1}{3} x[n] + \frac{4}{3} x[n-1] + \frac{1}{3} x[n-2] + y[n-2]
   \]
Assume that $y[n] = 0$, $n < 0$.

(a) (5 points) Determine the transfer functions ($z$-transforms) and the poles and zeros for each of the above integrators.

Running sum:

$$H(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

with a zero at $z = 0$ and a pole at $z = 1$.

Trapezoid rule:

$$H(z) = \frac{1}{2} \left( \frac{1 + z^{-1}}{1 - z^{-1}} \right), \quad |z| > 1$$

with a zero at $z = -1$ and a pole at $z = 1$.

Simpson’s rule:

$$H(z) = \frac{1}{3} \left( \frac{1 + 4z^{-1} + z^{-2}}{1 - z^{-2}} \right), \quad |z| > 1$$

with zeros at $z = -2 \pm \sqrt{3}$ and poles at $z = \pm 1$.

(b) (5 points) Determine and sketch the impulse response, $h[n]$, for $0 \leq n \leq 6$ for each of the above.

Running sum:

$$h[n] = 1, \quad 0 \leq n \leq 6$$

Trapezoid rule:

$$h[n] = \begin{cases} 1/2, & n = 0 \\ 1, & 1 \leq n \leq 6 \end{cases}$$

Simpson’s rule:

$$h[n] = \begin{cases} 1/3, & n = 0 \\ 2/3, & n = 2, 4, 6 \\ 4/3, & n = 1, 3, 5 \end{cases}$$

(c) (5 points) Suppose we switch the roles of $x[n]$ and $y[n]$ so that now $x[n]$ is the output and $y[n]$ is the input. Determine the three resulting new transfer functions under this interpretation. In English, what would you call the operation(s) that these resulting systems perform?

The three transfer functions are respectively the inverses of the above three transfer functions. For causal systems, the ROCs are defined according to the new poles and are such that $|z| > \alpha_d$ where $\alpha_d$ is the outermost pole. These transfer functions correspond to systems that essentially are forms of integrators.

Problem # 2 (15 points)

The “center of gravity” of a sequence $x[n]$ can be defined as:

$$c = \frac{\sum_{n=-\infty}^{\infty} nx[n]}{\sum_{n=-\infty}^{\infty} x[n]}$$

Express $c$ in terms of $X(e^{j\omega})$, which is the Fourier transform of $x[n]$. Justify your answer by providing an accurate but concise derivation.
First note that
\[ X(e^{j\theta}) = \sum_n x[n] \]

Then, note that
\[ \mathcal{F}\{nx[n]\} = j \frac{dX(e^{j\omega})}{d\omega} = \sum_n nx[n]e^{-j\omega n} \]

so that
\[ j \frac{dX(e^{j\omega})}{d\omega} \bigg|_{\omega=0} = \sum_n nx[n] \]

giving
\[ c = j \frac{dX(e^{j\omega})}{d\omega} \bigg|_{\omega=0} \]

Problem # 3 (20 points)

In the following figure, \( h[n] \) is the impulse response of the LTI system within the inner box. The input to the system \( h[n] \) is \( v[n] \), and the output is \( w[n] \).

\[ x[n] \rightarrow h[n] \rightarrow w[n] \]

The z-transform of \( h[n] \), \( H(z) \), exists in the following region of convergence:
\[ 0 < r_{\min} < |z| < r_{\max} < \infty \]

(a) (6 points) Can the LTI system with impulse response \( h[n] \) be BIBO (bounded input yields bounded output) stable? If so, determine inequality constraints on \( r_{\min} \) and \( r_{\max} \) such that it is stable. If not, briefly explain why.

For the system to be BIBO stable, we must have that the ROC contains the unit circle, so that \( 0 < r_{\min} < 1 < r_{\max} < \infty \).

(b) (6 points) Is the overall system (in the large box, with input \( x[n] \) and output \( y[n] \)) LTI? If so, find its impulse response, \( g[n] \). If not, briefly explain why.
Look at each in turn.

\[ v[n] = \alpha^{-n} x[n] \]

\[ w[n] = \sum_k h[k]\alpha^{-(n-k)}x[n-k] \]

and

\[ g[n] = \alpha^n w[n] = \sum_k (h[k]\alpha^k) x[n-k] \]

So the underlying input/output relationship is just convolution, so the system is LTI.

When \( x[n] = \delta[n] \), we can see that the impulse response is the term in parentheses in the convolution sum above, and is \( g[n] = \alpha^n h[n] \).

(c) (8 points) Can the overall system be BIBO stable? If so, determine inequality constraints relating \( \alpha \), \( r_{\min} \), \( r_{\max} \) such that it is stable. If not, briefly explain why.

We know that

\[ H(z) = \sum_n h[n]z^{-n} \]

exists for \( r_{\min} < |z| < r_{\max} \). Looking at the entire system response,

\[ G(z) = \sum_n g[n]z^{-n} = \sum_n h[n](\alpha^{-1}z)^{-n} \]

we can see that \( G(z) \) exists for \( r_{\min} < |\alpha^{-1}z| < r_{\max} \) which means that the ROC for \( G(z) \) is \( |\alpha|r_{\min} < |z| < |\alpha|r_{\max} \) and for BIBO stability, we need \( |\alpha|r_{\min} < 1 < |\alpha|r_{\max} \).

Problem # 4 (15 points)

Let \( x[n] = \alpha^{ln} \), where \( \alpha \) is real.

Is the Fourier transform of \( x[n] \) conjugate symmetric or conjugate anti-symmetric or neither? You must justify your answer by showing the work that leads you to your conclusion.

\( x[n] \) is real, so as a sanity check we can look at table 2.1 on page 56 in the book which says that any real \( x[n] \) has a conjugate symmetric Fourier Transform (i.e., that \( X(e^{j\omega}) = X^*(e^{-j\omega}) \).

We have to demonstrate this, however. There are many possible valid ways to show this. One particularly simple way is as follows.

\[ X^*(e^{-j\omega}) = \left( \sum_n \alpha^{ln}e^{-j(\omega)n} \right)^* \]

\[ = \sum_n \alpha^{ln} (e^{j\omega n})^* \]

\[ = \sum_n \alpha^{ln} e^{j\omega n} \]

\[ = X(e^{j\omega}) \]

Problem # 5 (25 points)

We are interested in sampling the real signal \( x(t) \) which is band-limited to \( \pm \Omega_N \). The continuous-time Fourier transform \( X(j\Omega) \) of \( x(t) \) is shown below. Note that this shows both the real part of \( X(j\Omega) \) (which is notated by \( \text{Re} \{X(j\Omega)\} \))
and the imaginary part (notated by $Im\{X(j\Omega)\}$).

We propose the following two sampling strategies.

1. **Strategy 1**: Sampling $x(t)$ at the Nyquist rate, to obtain $x_1[n]$.

   $$x_1[n] = x(nT)$$

   where $T = \pi/\Omega_N$.

2. **Strategy 2**: First filtering $x(t)$ with

   $$H(j\Omega) = \begin{cases} 1, & \Omega \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

   which is a filter that removes negative frequencies as shown below. Since $x(t)$ is real, its Fourier transform has conjugate symmetry and therefore no “information” is lost in this filtering process. Next, the filtered signal, say $f(t)$, is sampled at the new Nyquist rate.

   $$x_2[n] = f(nT')$$

   where $T' = 2\pi/\Omega_N$.

(a) (20 points) Show that all information is captured in the second sampling scheme by designing a fully discrete-time system for recovering $x_1[n]$ from $x_2[n]$. 
A discrete-time system to recover \( x_1[n] \) from \( x_2[n] \) is first upsample by 2, filter with

\[
H(e^{j\omega}) = \begin{cases} 
4, & 0 \leq \omega < \pi \\
0, & -\pi \leq \omega < 0
\end{cases}
\]

and then take the real part of the result. This is indicated in the following figure (along with the frequency domain view of what is happening):

(b) (5 points) What are some advantages and disadvantages associated with each sampling scheme?

**Scheme # 1:**
Disadvantages: higher sampling rate
Advantages: 1) needs only 1 A/D converter, 2) is relatively simple.

**Scheme # 2:**
Disadvantages: 1) Needs 2 A/D converters (for the real and imaginary part) 2) more complicated filtering involved, 3) introduces a delay, 4) hard to realize the filters.
Advantages: lower sampling rate.

Note that both methods need the same amount of information to represent \( x(t) \). Scheme 1 needs \( \Omega_N/\pi \) real samples/sec but scheme 2 needs \( \Omega_N/2\pi \) complex samples/sec.

Problem # 6 (10 points)
Consider a causal and stable (but otherwise unknown) system with input \( x[n] = e^{j\frac{\Omega_0}{2\pi}n} \) and \( y[n] = 3e^{j\frac{\Omega_0}{2\pi}n} \).

(a) (1 point) Could the system be LTI? Explain your reasoning.

Yes, it could be. For example, \( H(z) = 3 \) which is LTI is one example.

(b) (3 points) If you answered yes to part (a), give a possible system function, \( H(z) \), for the system.

As given in the example above, \( H(z) = 3 \) is a LTI system that works.

(c) (3 points) If you answered yes to part (a), is there more than one LTI system consistent with the input-output pair?

Yes, there is more than one possible such LTI system. Any LTI system with \( H(z)|_{z=e^{j\frac{\Omega_0}{2\pi}n}} = 3 \) would work.

(d) (3 points) If you answered yes to part (a), is the system guaranteed to be LTI?

The system is not guaranteed to be (i.e., it need not be) LTI. Consider a system whose output is \( 3e^{j\frac{\Omega_0}{2\pi}n} \) regardless of its input. This system is clearly not linear.