Submodularity beyond submodular energies: Coupling edges in graph cuts

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local pairwise random fields ...
Markov Random Fields and Energies

\[
p(x | z) \propto \exp(-E_\Psi(x; z))
\]

MAP \quad x^* = \arg \min_x E_\Psi(x; z)
Markov Random Fields and Energies

\[ p(x \mid z) \propto \exp(-E_{\psi}(x; z)) \]

\[ \text{MAP} \quad x^* = \arg \min_x E_{\psi}(x; z) \]

\[ E(x; z) = \sum_i \psi_i(x_i) + \sum_{(i,j) \in \mathcal{N}} \psi_{ij}(x_i, x_j) \]
\[ p(x \mid z) \propto \exp(-E_\Psi(x; z)) \]

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\[
E(x; z) = \sum_{e \in \Gamma \times \mathfrak{E}_t} w_e + \sum_{e \in \Gamma \times \mathfrak{E}_n} w_e
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Markov Random Fields and Energies

\[ p(x | z) \propto \exp(-E_\psi(x; z)) \]

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\[ E(x; z) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e + \sum_{e \in \Gamma \cap \mathcal{E}_n} w_e \]
Graph Cuts

Cooperative Cuts

Optimization

Applications
Couple edges globally.
Couple edges globally.
Couple edges globally
Richer Cuts: Cooperative Cuts

\[ E(x) = \sum_{e \in \Gamma x} w(e) = w(\Gamma x) \]
Richer Cuts: Cooperative Cuts

\[ E(x) = \sum_{e \in \Gamma x} w(e) \]
\[ = w(\Gamma x) \]

\[ E_f(x) = f(\Gamma x) \]
submodular function on edges
Richer Cuts: Cooperative Cuts

\[ E(x) = \sum_{e \in \Gamma x} w(e) = w(\Gamma x) \]

\[ E_f(x) = f(\Gamma x) \]

submodular function on edges

non-submodular &
global energy
Coupling via Submodularity
Coupling via Submodularity

\[ f(A \cup e) - f(A) \geq f(A \cup B \cup e) - f(A \cup B) \]

- Graph Cuts: LHS = RHS
  “it does not matter which other edges are cut”
Coupling via Submodularity

\[ f(A \cup e) - f(A) \geq f(A \cup B \cup e) - f(A \cup B) \]

- Graph Cuts: LHS = RHS
  “it does not matter which other edges are cut”

submodularity:
- reward co-occurrence
- structure
Special cases of cooperative cuts:

- (robust) $P^n$ potentials (Kohli et al. ’07,’09)
- label costs (Delong et al. ’11)
- discrete versions of norm-based cuts (Sinop & Grady ’07)
- ...
Optimization?

Theorem

Minimum Cooperative Cut is NP-hard.
Optimization?

\[(s, t)\text{-cut } \Gamma \subseteq \mathcal{E} \text{ with min cost } f(\Gamma)\].

**Theorem**

*Minimum Cooperative Cut is NP-hard.*
\[ \Gamma_0 = \emptyset; \]

**repeat**

compute upper bound \( \hat{f}_i \geq f \) based on \( \Gamma_{i-1} \);

**until** convergence;

\[ \hat{f}_i(\Gamma_{i-1}) = f(\Gamma_{i-1}) \]
\[ \Gamma_0 = \emptyset; \]
\begin{algorithmic}
  \State \textbf{repeat}
  \State \hspace{1em} compute upper bound \( \hat{f}_i \geq f \) based on \( \Gamma_{i-1} \);
  \State \hspace{1em} \( \Gamma_i \in \text{argmin} \{ \hat{f}_i(\Gamma) \mid \Gamma \text{ a cut} \} \); \hspace{1em} // \hspace{1em} \text{Min-cut!}
  \State \hspace{1em} \( i = i + 1; \)
  \State \textbf{until} convergence;
\end{algorithmic}

\[ \hat{f}_i(\Gamma_{i-1}) = f(\Gamma_{i-1}) \]
\[
\Gamma_0 = \emptyset; \\
\text{repeat} \\
\text{compute upper bound } \hat{f}_i \geq f \text{ based on } \Gamma_{i-1}; \\
\Gamma_i \in \arg\min \{ \hat{f}_i(\Gamma) \mid \Gamma \text{ a cut} \}; \quad \text{// Min-cut!} \\
i = i + 1; \\
\text{until convergence ;}
\]

Worst-case approximation bound:

\[
E_f(x) \leq \frac{|\Gamma^*|}{1+|\Gamma^*|} E_f(x^*) \quad \text{for } \nu = \frac{\min_{e \in \Gamma^*} \rho_e(E \setminus e)}{\max_{e \in C^*} f(e)}
\]
Image Segmentation

Random Walker  
Curvature reg.  
Graph Cut
Image Segmentation

prefer congruous boundaries
Selective Discount for Congruous Boundaries

\[ E_w(x) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e + \lambda \sum_{e \in \Gamma \cap \mathcal{E}_n} w_e \]

\[ E_f(x) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e + \lambda f(\Gamma \cap \mathcal{E}_n) \]
Selective Discount for Congruous Boundaries

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\[ E_f(x) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e + \lambda f(\Gamma \cap \mathcal{E}_n) \]

- discount for co-occurring similar edges
- no discount for dissimilar edges
Structured Discounts

groups $S_i$ of edges

$$f(\Gamma) = \sum_i f_i(\Gamma \cap S_i)$$
Structured Discounts

$$f(\Gamma) = \sum_i f_i(\Gamma \cap S_i)$$

groups $$S_i$$ of edges
Structured Discounts

$$f(\Gamma) = \sum_i f_i(\Gamma \cap S_i)$$

groups $S_i$ of edges
Some Results: Shading

Graph Cut
7.39%

CoopCut
2.23%

7.65%
3.50%
## Some Results: Shading

<table>
<thead>
<tr>
<th>Method</th>
<th>discount</th>
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<th>color</th>
<th>high-freq</th>
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<tbody>
<tr>
<td>Graph Cut: no discount</td>
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<td>14.03</td>
<td>3.41</td>
<td>2.56</td>
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<tr>
<td>CoopCut (1 group): discount</td>
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<td>11.58</td>
<td>2.95</td>
<td>1.49</td>
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<tr>
<td>CoopCut (15 groups): structured discount</td>
<td>3.63</td>
<td>1.69</td>
<td>1.27</td>
<td></td>
</tr>
</tbody>
</table>

- Graph Cut: 5.08%
- CoopCut: 0.64%
Shrinking bias

\[
\sum_{i} \psi_i(x_i) + \lambda \sum_{e \in E} \mathcal{W}_e
\]
Shrinking bias

\[
\sum_{i} \psi_i(x_i) + \lambda \sum_{e \in \Gamma X} w_e
\]

Graph Cut
Shrinking bias

\[
\sum \psi_i(x_i) + \lambda f(\Gamma x)
\]

CoopCut

Graph Cut
Shrinking bias

\[ \sum_{i} \psi_i(x_i) + \lambda f(\Gamma x) \]

CoopCut

Graph Cut
Summary: Coupling Edges in Graph Cuts

- global, non-submodular family of energies
- NP-hard, but...
  - graph structure
  - indirect submodularity
  → efficient approximation algorithm

- applications
  - guide segmentations via edge coupling