Submodularity and Big Data

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Outline

1. Big Data - The Good, The Bad, and The Ugly
2. Submodularity: Generalized Independence
3. Document Summarization
4. Speech Summarization
5. General Summarization
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1. Big Data - The Good, The Bad, and The Ugly
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Big Data

- Big Data is Big and Getting Bigger.

Every day, we create 2.5 quintillion bytes (2.5 billion gigabytes) of data (source: IBM).
- 90% of the world’s data has been created in the last two years.
Big data is different, emergent properties, a higher-level gestalt. Big data is big, it requires enormous compute to process that much data. Big data is overabundant, more samples than needed.
“More is Different”, P.W. Anderson, 1972, “The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe.”

Bigger is not just bigger — bigger is different.

Emergent properties: the whole is not just more but very different than the sum of its parts:

- H2O molecules: small ($n$-body), medium (water), large (global weather systems), all w/o changing form of molecular collision events.
- Neurons: small (neural spike trains, population coding), large (biological intelligence, consciousness)
- Pixels: small (random dot patterns), large (pointillism, photograph, visual scenes).
- Sinusoids: small (harmonic series), large (musical textures)
- People: small (psychology), large (collective intelligence, social choice theory, wisdom of the crowd).
More is indeed different, and this can be exploited. Organizations adopting “data-driven decision making” achieve productivity gains 5 to 6 percent (NYTs, 2/11/2012).
Big Data: The Bad

- Big Data is Very Big \(\Rightarrow\) Information Overload.
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- Personal information overload, even in 1970.

In a situation of information overload, inputs exceed the decision maker’s capacity to assimilate and act on the information as well as his/her ability to evaluate every alternative. (J. Walker, 1971)
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- Big data — addressing the new computational challenges:
  1. systems programming,
  2. parallel and distributed computing,
  3. efficient databases.
Many descriptions of big data
Big data is big in many ways
Big data is being over-hyped

- New York Times, “What You’ll Do Next,” by David Brooks,

Although, nice rebuttal by Andrew McAfee, “Pundits: Stop Sounding Ignorant About Data”, April 2013, Harvard Business Review.
Big Data: The Ugly

- Often, Big Data $\Rightarrow$ Big Redundancy

![Cartoon Image](image_url)
Big Data: The Ugly

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I started a task force to eliminate redundancies in our internal processes.

Really? I'm doing the same thing.

- Information Overload
Big Data: The Ugly

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- Information Overload $\Rightarrow$ Sample Overload
Big Data: The Ugly

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![Cartoon Image: Dilbert](image.png)

- Information Overload $\Rightarrow$ Sample Overload

- More samples (e.g., documents, photos, web pages, sentences, utterances, & facebook friends) than necessary to represent an amount of information.
Big Data in Machine Learning

Training and test set sizes are getting big, which is great!
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Automatic Speech Recognition
(Riccardi & Hakkani-Tür, 2005)
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**Machine Translation (MT)**
(Callison-Burch & Bloodgood, 2010)
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Automatic Speech Recognition (Riccardi & Hakkani-Tür, 2005)

Machine Translation (MT) (Callison-Burch & Bloodgood, 2010)

Diminishing Returns: the more you have, the less valuable is anything you don’t have. Bad for complex systems (e.g., deep models, SVMs).
Big Data: Summary

**The Good**

Big data is different, emergent properties, a higher-level gestalt.

**The Bad**

Big data is big, it requires enormous compute to process that much data.

**And the Ugly**

Big data is overabundant, more samples than needed.
Can statistical predictions be cost effective using small data?
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Goal: Extract core data from big data. Core data has the properties:

1. Retain information (i.e., decisions based on analysis of original must be close to analysis of core)

2. Same codebook as source. E.g., for text, reduced form of documents still use grammatical human-language sentences. For pictures, still a set of images. For training data, a subset.

3. Need not reconstruct original (need not decompress based on summary to reconstruct original).

4. Small core data should be small enough to significantly reduce cost.

Core data can be like a summary, cliff-notes, extractive summarization.
The Data Subset Selection Problem (DSSS)

- We have big finite set $V$ of size $n = |V|$ of elements.
- Select a small subset $A \subseteq V$ usable as a surrogate for $V$.
- Ultimate Goal: any question asked based on $V$ can be accurately answered based only on $A$.
- Ex: Extractive Document Summarization: $V$ is a set of sentences from multiple documents.
- Ex: Automatic speech recognition (ASR): $V$ is a set of training samples (e.g., a large set of acoustic utterances for ASR).
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Quality Functions and Costs

- A quality function \( f : 2^V \rightarrow \mathbb{R}_+ \).
- Given \( A \subseteq V \), the value \( f(A) \) measures the “quality” of or information within \( A \) (how good we measure \( A \) to be).
- Moreover, we might have a cost associated with each \( v \in V \) measured by a cost function \( c : V \rightarrow \mathbb{R}_+ \).
- Example: \( c(v) \) might be the length, or number of words, pixels, or complexity of element \( v \), and \( c(A) = \sum_{a \in A} c(a) \).
- If \( c(v) = 1 \) for all \( v \in V \), then \( c(A) = |A| \), the size of set \( A \).
Two sensible optimization strategies:

The “best within a given budget $b$” approach:

$$A^* \in \arg \max_{A \subseteq V : c(A) \leq b} f(A)$$  \hspace{1cm} (1)

The “least costly with a quality guarantee” approach:

$$A^* \in \arg \min_{A : f(A) \geq \alpha} c(A)$$  \hspace{1cm} (2)
Without making further assumptions about \( f \), the two optimization problems have exponential cost, even to approximate with any degree of quality assurance, independent of the \( P \neq NP \) question.
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When $f$ is monotone non-decreasing submodular, then the problems can be solved using the simple greedy algorithm with constant factor guarantees.
Maximization of Non-Decreasing Submodular Functions

- The problem is in general NP-hard (reduction from max-cut).
- Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions can be maximized, under a cardinality constraint, using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat until $|S_i| = k$:
  \begin{equation}
  S_{i+1} = S_i \cup \left\{ \arg\max_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{3}
  \end{equation}
- Has guarantee $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S) \approx 0.63 f(S)$.
- Depending on the “curvature” of $f$, this bound is much better still (better as $f$ becomes less curved).
- Feige (1998): can’t be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.
- Minoux (1977). Accelerated greedy algorithm significant speeds up process $O(n \log n)$ w. no further approximation needed.
A function $f$ is submodular if $\forall A, B \subseteq V$

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (4)$$

Equivalently, a function $f$ is submodular if it satisfies diminishing returns: for all $A \subseteq B$ and $v \notin B$,

$$f(A + v) - f(A) \geq f(B + v) - f(B) \quad (5)$$

In words: the value of $v$ diminishes as the context in which it is considered grows.

If a function is also normalized ($f(\emptyset) = 0$), and monotone non-decreasing ($f(A) \leq f(B)$ whenever $A \subseteq B$), then the function is said to be a “polymatroid”
Costs to a consumer are submodular. For example:
Ex. Submodular: Consumer Costs of Living

Costs to a consumer are submodular. For example:

\[ f(\text{McDonald's}) + f(\text{Fries}) \geq f(\text{Fries}) + f(\text{McDonald's}) \]
Costs to a consumer are submodular. For example:

\[ f(\text{coffee, fries}) + f(\text{hamburger}) \geq f(\text{coffee}) + f(\text{fries}) \]

When seen as diminishing returns:

\[ f(\text{coffee, fries}) - f(\text{fries}) \geq f(\text{coffee}) - f(\text{fries}) \]
Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
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- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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\vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
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1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
\]

- Let \( A = \{1, 2, 3\} \), \( B = \{3, 4, 5\} \), \( C = \{6, 7\} \), \( A_r = \{1\} \), \( B_r = \{5\} \).
- Then \( r(A) = 3 \), \( r(B) = 3 \), \( r(C) = 2 \).
- \( r(A \cup C) = 3 \), \( r(B \cup C) = 3 \).
- \( r(A \cup A_r) = 3 \), \( r(B \cup B_r) = 3 \), \( r(A \cup B_r) = 4 \), \( r(B \cup A_r) = 4 \).
- \( r(A \cup B) = 4 \), \( r(A \cap B) = 1 < r(C) = 2 \).
Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
\]

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- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
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- $r(A \cup A_r) = 3$, $\textcolor{red}{r(B \cup B_r) = 3}$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$. 
Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
| & | & | & | & | & | & | & | \\
| & | & | & | & | & | & | & | \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{pmatrix}
$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$. 
Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
$$

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.

Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.

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$r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.

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Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
| & | & | & | & | & | & | & | \\
| & | & | & | & | & | & | & |
\end{pmatrix}
$$

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.

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Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}.$

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\
2 & 0 & 3 & 0 & 0 & 0 & 4 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid \\
\mid & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid \\
\end{pmatrix}
$$

- Let $A = \{1, 2, 3\}, \ B = \{3, 4, 5\}, \ C = \{6, 7\}, \ A_r = \{1\}, \ B_r = \{5\}.$
- Then $r(A) = 3, \ r(B) = 3, \ r(C) = 2.$
- $r(A \cup C) = 3, \ r(B \cup C) = 3.$
- $r(A \cup A_r) = 3, \ r(B \cup B_r) = 3, \ r(A \cup B_r) = 4, \ r(B \cup A_r) = 4.$
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\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
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**Example: Rank function of a matrix**

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.

Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.

- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.
- $6 = r(A) + r(B) > r(A \cup B) + r(A \cap B) = 5$.
The Venn and Art of Submodularity

\[ r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \]

\[ = r(A_r) + 2r(C) + r(B_r) \]

\[ = r(A_r) + r(C) + r(B_r) \]

\[ = r(A \cap B) \]
Suppose we have a vector space where vectors represent data items (e.g., photos, documents, etc.), and span within the vector space represents “information”

Let the set $V$ represent the set of “big data”.

If we can find a subset $A \subseteq V$ such that $r(A) = r(V)$ then items $A$ span everything, and anything else not in $A$ is irrelevant given $A$.

Rank of a matrix (even in high dimensions) is not ideal for the big-data problem.
Shannon Information: Entropy

- Given $V = \{1, 2, \ldots, n\}$ and random variables $X_1, X_2, \ldots, X_n$.
- For a given subset $\{a_1, a_2, \ldots, a_k\} = A \subset V$ define the function
  \[ f(A) = H(X_{a_1}, X_{a_2}, \ldots, X_{a_k}) \triangleq H(X_A) \quad (6) \]
- $f(V)$ is the amount of Shannon information in all variables.
- Given $A$ such that $f(A) = f(V)$ then, then random variables $X_A$ contain all the information. Any random variable not in $A$ is a deterministic function of those in $A$.
- The entropy function is also submodular:
  \[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (7) \]
- Captures not only notion of dependence/independence but also partial dependence/independence.
- Entropy is hard to evaluate in practice (requires a joint distribution).
Polymatroids: Generalized Dependence

- All submodular functions express “abstract independence”
Polymatroids: Generalized Dependence

- All submodular functions express “abstract independence”
- Within some combinatorial geometry defined by a polymatroid function, there is a notion of “independence”, i.e., $A \perp B$:

$$f(A \cup B) = f(A) + f(B),$$  \hspace{1cm} (8)
Polymatroids: Generalized Dependence

- All submodular functions express “abstract independence”
- Within some combinatorial geometry defined by a polymatroid function, there is a notion of “independence”, i.e., $A \perp B$:
  \[
  f(A \cup B) = f(A) + f(B),
  \]
  (8)
- and a notion of “conditional independence”, i.e., $A \perp B \mid C$:
  \[
  f(A \cup B \cup C) + f(C) = f(A \cup C) + f(B \cup C)
  \]
  (9)
Polymatroids: Generalized Dependence

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(9)

- and a notion of “dependence” (conditioning reduces valuation):

$$f(A|B) \triangleq f(A \cup B) - f(B) < f(A),$$

(10)
Polymatroids: Generalized Dependence

- All submodular functions express “abstract independence”
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(8)

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(9)

- and a notion of “dependence” (conditioning reduces valuation):

$$f(A|B) \triangleq f(A \cup B) - f(B) < f(A),$$

(10)

- and a notion of “conditional mutual information

$$I_f(A; B|C) \triangleq f(A \cup C) + f(B \cup C) - f(A \cup B \cup C) - f(C) \geq 0$$
Polymatroids: Generalized Dependence

- All submodular functions express “abstract independence”
- Within some combinatorial geometry defined by a polymatroid function, there is a notion of “independence”, i.e., $A \perp \perp B$:
  \[ f(A \cup B) = f(A) + f(B), \]  
  (8)
- and a notion of “conditional independence”, i.e., $A \perp B \mid C$:
  \[ f(A \cup B \cup C) + f(C) = f(A \cup C) + f(B \cup C) \]  
  (9)
- and a notion of “dependence” (conditioning reduces valuation):
  \[ f(A \mid B) \triangleq f(A \cup B) - f(B) < f(A), \]  
  (10)
- and a notion of “conditional mutual information
  \[ I_f(A; B \mid C) \triangleq f(A \cup C) + f(B \cup C) - f(A \cup B \cup C) - f(C) \geq 0 \]
- and a notion of “information amongst a collection of sets”
  \[ I_f(S_1; S_2; \ldots; S_k) = \sum_{i=1}^{k} f(S_k) - f(S_1 \cup S_2 \cup \cdots \cup S_k) \]  
  (11)
Many set functions are submodular, including:

- Graph cut functions, hypergraph cut functions
- Facility location functions (in operations research)
- Value of information
- Joint Active Learning/Semi-Supervised Learning
- Social network influence, value of a friend.
- Many energy functions in graphical models and probabilistic reasoning
- Coverage functions (set cover, sensor placement, . . .)
- Economies of scale, network externalities, manufacturing costs

Submodular functions can be minimized in polynomial time!

As previously mentioned, submodular functions can be constant-factor approximately optimized in low-order polynomial time.
Submodular approaches to Big Data Subset Selection

- We are given a big data set $V$.
- Approach: find a good polymatroid function $f : 2^V \rightarrow \mathbb{R}_+$ that represents information in $V$.

1) Heuristic: build $f$ by hand, hoping that $f$ is a good proxy for the information within $V$. Acknowledge that $f$ is a surrogate objective, guarantees are only in terms of $f$.

2) More recent approach: attempt to learn $f$ in some fashion based on training data.

- We report on both kinds of results next.
Outline

1. Big Data - The Good, The Bad, and The Ugly
2. Submodularity: Generalized Independence
3. Document Summarization
4. Speech Summarization
5. General Summarization
We extract sentences (green) as a summary of the full document.
Extractive Document Summarization

- We extract sentences (green) as a summary of the full document

  [Extracted sentences shown as green bars]

  The summary on the left is a subset of the summary on the right. Consider adding a new (blue) sentence to each of the two summaries. The marginal (incremental) benefit of adding the new (blue) sentence to the smaller (left) summary is no more than the marginal benefit of adding the new sentence to the larger (right) summary. 

  diminishing returns ⇔ submodularity
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- diminishing returns $\Leftrightarrow$ submodularity
Problem setup

- The ground set $V$ corresponds to all the sentences in a document.
- Extractive document summarization: select a small subset $S \subseteq V$ that accurately represents the entirety (ground set $V$).
- The summary $S^*$ must be (budget) length-limited.

$$c(S^*) = \sum_{i \in S^*} c_i \leq b$$  \hspace{1cm} (12)

- A set function $f : 2^V \rightarrow \mathbb{R}$ measures the quality of the summary $S$.

Problem (Document Summarization Optimization Problem)

$$S^* \in \arg\max_{S \subseteq V} f(S) \text{ subject to: } \sum_{i \in S} c_i \leq b.$$  \hspace{1cm} (13)
When \( f \) is both **monotone** and **submodular**:

- Greedy algorithm with partial enumeration (Sviridenko, 2004), theoretical guarantee of near-optimal solution, but not practical.
- A greedy algorithm (Lin and Bilmes, 2010): near-optimal with theoretical guarantee \( (1 - 1/\sqrt{e}) \), and practical/scalable!
  - Choose next element with largest ratio of gain over **scaled** cost:

\[
  k \leftarrow \argmax_{i \in U} \frac{f(G \cup \{i\}) - f(G)}{(c_i)^r}.
\]

- Scalability: the \( \argmax \) above can be solved by \( O(\log n) \) calls of \( f \), thanks to submodularity
- Integer linear programming (ILP) takes 17 hours vs. greedy which takes < 1 second!!
The General Form of Our Submodular Functions

- Two properties of a good summary: relevance and non-redundancy.
- The redundancy penalty is usually what violates monotonicity.
- Our approach: we positively \textit{reward diversity} instead of negatively penalizing redundancy:

\[ f(S) = \mathcal{L}(S) + \lambda \mathcal{R}(S) \]

- \( \mathcal{L}(S) \) measures the coverage (or fidelity) of summary set \( S \) to the document.
- \( \mathcal{R}(S) \) rewards diversity in \( S \).
- \( \lambda \geq 0 \) is a trade-off coefficient.
Coverage function

Covered Function

\[ \mathcal{L}(S) = \sum_{i \in V} \min \{ C_i(S), \alpha C_i(V) \} \]

- \( C_i \) measures how well \( i \) is covered by \( S \).
- One simple possible \( C_i \) (that we use) is:

\[ C_i(S) = \sum_{j \in S} w_{i,j}, \]

where \( w_{i,j} \geq 0 \) measures the similarity between \( i \) and \( j \).
- With this \( C_i \), \( \mathcal{L}(S) \) is monotone submodular, as required.
Diversity reward function

\[ R(S) = \sum_{i=1}^{K} \sqrt{\sum_{j \in P_i \cap S} r_j}. \]

- \( P_i, i = 1, \ldots, K \) is a partition of the ground set \( V \)
- \( r_j \geq 0 \): singleton reward of \( j \), which represents the importance of \( j \) to the summary.
- square root over the sum of rewards of sentences belong to the same partition (diminishing returns).
- \( R(S) \) is monotone submodular as well.
Alternatively, we can utilize multiple partitions/clusterings, produce a diversity reward function for each one, and mix them together.

\[
\mathcal{R}(S) = \lambda_1 \sum_{i=1}^{K_1} \sqrt{\sum_{j \in P_i^{(1)} \cap S} r_j} + \lambda_2 \sum_{i=1}^{K_2} \sqrt{\sum_{j \in P_i^{(2)} \cap S} r_j} + \cdots
\]
DUC Evaluations

- DUC (Document Understanding Conference) data
  http://duc.nist.gov/


- Tasks are both query independent (DUC '04) and query dependent summarization (DUC '05-'07), which is more like web search.

- Standard measure of evaluation performance is the ROUGE measure.

- ROUGE is based on a collection of human generated summaries, so the ROUGE measure can be only used to evaluate a summary.
NIST’s ROUGE-N evaluation function

While NIST’s ROUGE-N recall score is the standard evaluation measure, it turns out also to be submodular:

\[
f_{\text{ROUGE-N}}(S) \triangleq \frac{\sum_{i=1}^{K} \sum_{e \in R_i} \min(c_e(S), r_{e,i})}{\sum_{i=1}^{K} \sum_{e \in R_i} r_{e,i}},
\]

where

- $S$ is the candidate summary (a set of sentences extracted from the ground set $V$).
- $c_e : 2^V \to \mathbb{Z}_+$ is the number of times an $n$-gram $e$ occurs in summary $S$, clearly a modular function for each $e$.
- $R_i$ is the set of $n$-grams contained in the reference summary $i$ (given $K$ reference summaries).
- and $r_{e,i}$ is the number of times $n$-gram $e$ occurs in reference summary $i$.
- ROUGE-N is of course unavailable to optimize directly.
Generic Summarization

Rouge-1: higher is better

- DUC-04: generic summarization

Table: ROUGE-1 recall (R) and F-measure (F) results (%) on DUC-04. DUC-03 was used as development set.

<table>
<thead>
<tr>
<th>DUC-04</th>
<th>R</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}_1(S)$</td>
<td>39.03</td>
<td>38.65</td>
</tr>
<tr>
<td>$\mathcal{R}_1(S)$</td>
<td>38.23</td>
<td>37.81</td>
</tr>
<tr>
<td>$\mathcal{L}_1(S) + \lambda \mathcal{R}_1(S)$</td>
<td>39.35</td>
<td>38.90</td>
</tr>
<tr>
<td>Takamura and Okumura (2009)</td>
<td>38.50</td>
<td>-</td>
</tr>
<tr>
<td>Wang et al. (2009)</td>
<td>39.07</td>
<td>-</td>
</tr>
<tr>
<td>Lin and Bilmes (2010)</td>
<td>-</td>
<td>38.39</td>
</tr>
<tr>
<td>Best system in DUC-04 (peer 65)</td>
<td>38.28</td>
<td>37.94</td>
</tr>
</tbody>
</table>
**DUC-05 results**

Rouge-2: higher is better

Table: ROUGE-2 recall (R) and F-measure (F) results (%)

<table>
<thead>
<tr>
<th>Formula</th>
<th>R</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1(S) + \lambda R_Q(S)$</td>
<td>7.82</td>
<td>7.72</td>
</tr>
<tr>
<td>$L_1(S) + \sum_{\kappa=1}^{3} \lambda_{\kappa} R_{Q,\kappa}(S)$</td>
<td>8.19</td>
<td>8.13</td>
</tr>
<tr>
<td>Daumé III and Marcu (2006)</td>
<td>6.98</td>
<td>-</td>
</tr>
<tr>
<td>Wei et al. (2010)</td>
<td>8.02</td>
<td>-</td>
</tr>
<tr>
<td>Best system in DUC-05 (peer 15)</td>
<td>7.44</td>
<td>7.43</td>
</tr>
</tbody>
</table>

- DUC-06 was used as training set for the objective function with single diversity reward.
- DUC-06 and 07 were used as training sets for the objective function with multi-resolution diversity reward.
DUC-06 results
Rouge-2: higher is better

<table>
<thead>
<tr>
<th>Equation</th>
<th>R</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}_1(S) + \lambda R_Q(S)$</td>
<td>9.75</td>
<td>9.77</td>
</tr>
<tr>
<td>$\mathcal{L}<em>1(S) + \sum</em>{\kappa=1}^{3} \lambda_\kappa R_{Q,\kappa}(S)$</td>
<td>9.81</td>
<td>9.82</td>
</tr>
<tr>
<td>Celikyilmaz and Hakkani-tür (2010)</td>
<td>9.10</td>
<td>-</td>
</tr>
<tr>
<td>Shen and Li (2010)</td>
<td>9.30</td>
<td>-</td>
</tr>
<tr>
<td>Best system in DUC-06 (peer 24)</td>
<td>9.51</td>
<td>9.51</td>
</tr>
</tbody>
</table>

- DUC-05 was used as training set for the objective function with single diversity reward.
- DUC-05 and 07 were used as training sets for the objective function with multi-resolution diversity reward
DUC-07 results
Rouge-2: higher is better

Table: ROUGE-2 recall (R) and F-measure (F) results (%)

<table>
<thead>
<tr>
<th>Formula</th>
<th>R</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1(S) + \lambda R_Q(S)$</td>
<td>12.18</td>
<td>12.13</td>
</tr>
<tr>
<td>$L_1(S) + \sum_{\kappa=1}^{3} \lambda_\kappa R_{Q,\kappa}(S)$</td>
<td><strong>12.38</strong></td>
<td><strong>12.33</strong></td>
</tr>
<tr>
<td>Toutanova et al. (2007)</td>
<td>11.89</td>
<td>11.89</td>
</tr>
<tr>
<td>Haghighi and Vanderwende (2009)</td>
<td>11.80</td>
<td>-</td>
</tr>
<tr>
<td>Celikyilmaz and Hakkani-tür (2010)</td>
<td>11.40</td>
<td>-</td>
</tr>
<tr>
<td>Best system in DUC-07 (peer 15), using web search</td>
<td><strong>12.45</strong></td>
<td>12.29</td>
</tr>
</tbody>
</table>

- DUC-05 was used as training set for the objective function with single diversity reward.
- DUC-05 and 06 were used as training sets for the objective function with multi-resolution diversity reward.
Max-Margin Learning of Submodular Mixtures

- Learning submodular functions is hard, but we have recently developed a method to learn mixtures of submodular components.
- I.e., We consider hypothesis functions with the following form

\[
h(x; w) = \arg\max_{y \in Y_x} s(x, y) \\
= \arg\max_{y \in Y_x} \sum_i w_i f_i(x, y).
\]

where \( w \geq 0 \) and \( f_i : \mathcal{X} \times Y_x \rightarrow \mathbb{R} \) is submodular on \( Y_x \) for a given \( x \in \mathcal{X} \).

- We learn the mixture coefficients using a submodular loss (related to rouge) and a large-margin learning objective.
- Submodular components consist of the types above, with addition of facility location like components.
DUC-04 Results
Rouge-1: higher is better

Lin & Bilmes, 2012
Lin & Bilmes, ACL 2011
Lin & Bilmes, NAACL 2010

ROUGE-1 Recall (%)
ROUGE-1 F-Measure (%)
DUC-05 Results
Rouge-2: higher is better

Lin & Bilmes, 2012
Lin & Bilmes, ACL 2011

DUC-05 best system
Fidelity+Diversity
Submodular mixture

ROUGE-2 Recall (%)
ROUGE-2 F-Measure (%)
DUC-06 Results

Rouge-2: higher is better

Lin & Bilmes, 2012

Lin & Bilmes, ACL 2011

ROUGE-2 Recall (%)
ROUGE-2 F-Measure (%)

DUC-06 best system
Fidelity+Diversity
Submodular mixture

Lin & Bilmes,
ACL 2011

Lin & Bilmes,
2012

ROUGE-2 Recall (%)
ROUGE-2 F-Measure (%)
DUC-07 Results
Rouge-2: higher is better

<table>
<thead>
<tr>
<th>ROUGE-2 Recall (%)</th>
<th>ROUGE-2 F-Measure (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.20</td>
<td></td>
</tr>
<tr>
<td>12.25</td>
<td></td>
</tr>
<tr>
<td>12.30</td>
<td></td>
</tr>
<tr>
<td>12.35</td>
<td></td>
</tr>
<tr>
<td>12.40</td>
<td></td>
</tr>
<tr>
<td>12.45</td>
<td></td>
</tr>
<tr>
<td>12.50</td>
<td></td>
</tr>
<tr>
<td>12.55</td>
<td></td>
</tr>
</tbody>
</table>

DUC-07 best system Fidelity+Diversity Submodular mixture

Lin & Bilmes, 2012

Lin & Bilmes, ACL 2011

J. Bilmes
Submodularity and Big Data
Outline

1. Big Data - The Good, The Bad, and The Ugly
2. Submodularity: Generalized Independence
3. Document Summarization
4. Speech Summarization
5. General Summarization
Speech Subset Selection: Two Forms

1. **Corpus Summarization**: Given a large set of speech utterances $V = \{v_1, v_2, \ldots, v_n\}$, choose a small subset $A \subseteq V$ that is representative of $V$.

- Summary must be representative (and answer any query accurately) relative to the whole.
- Goal: training on summary should result in same performance as training on whole.
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2. **Corpus Selection**: Given a large set of speech utterances \( V = \{v_1, v_2, \ldots, v_n\} \), choose a good subset \( A \subseteq V \) that limits the vocabulary size, say \( \Gamma(A) \).
   - Ex: large amount of acoustics while limit complexity of language model
Corpus Summarization: motivation

- Large vocabulary speech recognition training is both resource (disk, memory) and time consuming.
- Particularly acute with recent models (Deep Neural Networks) which can take weeks to train a single configuration of a single system.

Training (and test) data sets are redundant. From: Riccardi & Hakkani-Tur, 2005, although this kind of curve is typical.

- Why waste time/resources on information you already know?
Instantiating a Submodular Function: Some Choices

- Polymatroid: non-negative monotone non-decreasing submodular
- Facility location:

\[ f(A) = \sum_{i \in V} \max_{j \in A} w_{ij} \]  

(15)

- Saturated graph cut:

\[ f(A) = \sum_{i \in V} \min(C_i(A), \alpha C_i(A)) \]  

(16)

where \( C_i(A) = \sum_{j \in A} w_{ij}. \)

- Diversity function:

\[ f(A) = \sum_{k=1}^{K} \sqrt{m(A \cap V_k)} \]  

(17)

for partition \( V = V_1 \cup \cdots \cup V_k. \)
Results on TIMIT
Fisher Kernel (Submodular) vs. Histogram Entropy (Not-submodular)

Improvements of Fisher Kernels and Entropy Method

Relative Improvement (%)
Percentage of Speech (%)

-2.00%
0%
2.00%
4.00%
6.00%
2.5 5 10 20 30 40 50

Improvements of Fisher Kernels and Entropy Method

-2.00%
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Corpus Selection: motivation

- Machine learning: complexity is often linear in number of samples but polynomial in the number of types of objects.
- Canonical example: speech recognition: adding more training samples is relatively easy, except when the vocabulary expands (e.g., $O(N^3)$ or $O(N^4)$).
- This inhibits rapid turnaround time for novel and expensive surface methods (e.g., deep acoustic modeling in speech recognition).
- Goal: find a way to select a subset of the data while limiting the number of types.
Corpus Selection: description

- Bipartite graph \((V, F, E)\) where \(V\) is the set of utterances (sentences) and \(F\) is the set of words.
- \(\gamma(X)\) are the neighbors of \(X\), \(\zeta(Y)\) are the sole neighbors of \(Y\).
- \(\Gamma(X) \triangleq w(\gamma(X))\) is submodular, where \(w : 2^V \rightarrow \mathbb{R}\) is a modular weight function, while \(w(\zeta(Y))\) is supermodular.
- King et al. 2005: maximizing \(w(\zeta(Y))\) with cardinality constraint, using greedy algorithm. This can do unboundedly poorly.
Fast Parametric Max flow implementation

Figure: In subfigure (a), $V = \{v_1, v_2, v_3, v_4\}$ and $F = \{f_1, f_2, f_3\}$. For $X = \{v_3, v_4\}$, $\gamma(X) = \{f_2, f_3\}$; for $Y = \{f_1\}$, $\zeta(Y) = \{v_1\}$. In (b), the s-t graph corresponding to $w(V \setminus X) + \lambda \Gamma(X)$.
Results - try to get words with many phones

Corpus D, $\Gamma_2(X) = \sum_{i \in \gamma(X)} p_i = \sum_{i \in \gamma(X)} \frac{c}{q_i}$, where $c$ is a constant, and $q_i$ is the number of phonemes in the pronunciation of word $i$.

Figure: Venn diagram showing the vocabulary difference between SVitchboard-50 and D-50.
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Submodular class notes:
http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/

Submodularity in machine learning: http://melodi.ee.washington.edu/~bilmes/pgs/class_rescat.html#Submodularity


The End: Thank you!

Submodularity

Learn to:
- Choose your data sets with a $1 - 1/e$ guarantee!
- Minimize your functions in polynomial time!
- Draw beautiful polyhedra!
- Solve exponentially large linear programs in polynomial time!

Paul E. Matroid
Moniton Submodularanian

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$
Can Data Compression Help?

- Data is often redundant. When fully non-redundant, it is in compressed form.
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Properties of compression:

1. **Lossless compression** algorithms typically reconstruct exactly the original data, at the original size. Information about size is preserved.
2. **Lossy compression** also often preserves size on reconstruction (e.g., music or video).
3. **Compression often changes the code**, uses codewords that are uninterpretable (look like random bit strings) except in the context of a decoder. E.g., a human can’t directly read a gzipped .txt file.
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- Hence, compression has undesirable properties in this case.