Announcements

- HW1 was due last night.
- HW2 should be ready by Friday night (I’ll send email when ready).
Polymatroidal polyhedron (or a “polymatroid”)

Definition 2.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$

2. If $y \leq x \in P$ then $y \in P$ (called down monotone).

3. For any $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (called a $P$-basis of $x$), has the same component sum $y(E)$. That is for any two maximal vectors $y^1, y^2 \in P$, we have $y^1(E) = y^2(E)$.
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Another way of saying this is that a polymatroid is a compact set that is zero containing, down monotone, and any maximal vector $y$ in $P$, bounded by another vector $x$, has the same vector rank.
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- Another way of saying this is that a polymatroid is a compact set that is zero containing, down monotone, and any maximal vector $y$ in $P$, bounded by another vector $x$, has the same vector rank.
- Compare with the definition of a **matroid**: a set system that is empty-set containing, down closed, and any maximal set $I$ in $\mathcal{I}$, bounded by another set $A$, has the same matroid rank.
Polymatroid function and its polyhedron.

Definition 2.2

A polymatroid function is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P_f$ associated with a polymatroid function as follows

$$P_f = \left\{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \right\} \quad (1)$$

$$= \left\{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \right\} \quad (2)$$
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 2.3**

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}^E_+$, and any $P_f$-basis $y^x$ of $x$, we have that the component sum of $y$ is

$$\max (y(E) : y \leq x, y \in P_f) = y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E)$$

(3)

As a consequence, $P_f$ is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 2.3**

Let \( f \) be a polymatroid function defined on subsets of \( E \). For any \( x \in \mathbb{R}_+^E \), and any \( P_f \)-basis \( y^x \) of \( x \), we have that the component sum of \( y \) is

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(3)

As a consequence, \( P_f \) is a polymatroid.

Therefore, by taking elements \( E \setminus A \) to be zero in \( x \), we can define/recover the submodular function from the polymatroid polyhedron via the following:

\[
f(A) = \max \{y(A) : y \in P_f\}
\]

(4)
Consider the right hand side of the theorem:
\[ \min (x(A) + f(E \setminus A) : A \subseteq E) \]
Where are we going with this?

Consider the right hand side of the theorem:
\[
\min (x(A) + f(E \setminus A) : A \subseteq E) = \min (x(E \setminus A) + f(A) : A \subseteq E)
\]

We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
Matroid case

Considering the above theorem, the matroid case is now a special case, where we have that:

**Corollary 3.1**

We have that:

\[
\max \{ y(E) : y \in P_{\text{ind. set}}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\]  

(5)

where \( r_M \) is the matroid rank function of some matroid.
Most violated inequality problem

- Consider

\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \quad (6) \]
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\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \] (6)

We saw before that \( P_r = \text{Pind. set} \).

Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \notin P_r \).

The most violated inequality when \( x \) is considered w.r.t. \( P_r \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e., \( \max \{ x(A) - r_M(A) : A \subseteq E \} \geq \not A \).
Most violated inequality problem

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  \[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \tag{6} \]

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  \[ \max \{ x(A) - r_M(A) : A \subseteq E \} \]

- This corresponds to \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \). ✔
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\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \]  \hspace{1cm} (6)

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This corresponds to \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \).

More importantly, \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) a form of submodular function minimization, namely \( \min \{ r_M(A) - x(A) : A \subseteq E \} \) for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).
Approach

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}^E_+$;
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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. Of course, for any such $y$ we must have that $y(E) \leq r(A) + x(E \setminus A)$. 

By the above theorem, the existence of such an $A$ will certify that $x(E)$ is maximal in $P_{\text{ind. set}}$ and minimal in terms of $f(A) = r_M(A) - x(A)$. This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of $f$. This will also run in polynomial time.
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- This will also run in polynomial time.
Idea of the algorithm

- We build up $y$ from the ground up.
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- We keep a family of independent sets $(l_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{l_i}$. 
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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set.}}$, but also we’ll keep $y \leq x$.
- It’s going to take us a few lectures to fully develop this algorithm, so please keep mind of the overall goal.
Bipartite Matching

Consider a bipartite graph $G = (V, F, E)$ where left nodes are $V$, right nodes are $F$, and $E \subseteq V \times F$ are the only edges.
Bipartite Matching

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- A **matching** $A \subseteq E$ is a subset of edges such that no two edges are incident to the same vertex.
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A matching $A \subseteq E$ is a subset of edges such that no two edges are incident to the same vertex.

A node $j$ is matched in $A$ if $(j, k) \in A$ for some $k \in F$, and otherwise $j$ is called unmatched. Likewise for some $k \in F$. 
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- A node $j$ is matched in $A$ if $(j, k) \in A$ for some $k \in F$, and otherwise $j$ is called unmatched. Likewise for some $k \in F$.
- Given $A \subseteq E$, an alternating path $S$ (relative to $A$) is an (undirected) path of unique edges that are alternatively in $A$ and not in $A$. I.e., if $S = (e_1, e_2, \ldots, e_s)$ is an alternating path, then $S \setminus A \triangleq S_{1/2}$ where $S_{1/2}$ is either the odd or the even elements of $S$. 
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- An $A \subseteq E$ is an augmenting path if it is an alternating path between two unmatched vertices.
Bipartite Matching

- Given a matching \( A \subseteq E \) (which might be empty), we can increase the matching if we can find an augmenting path \( S \).
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The updated matching becomes \( A' = A \setminus S \cup S \setminus A = A \oplus S \), where \( \oplus \) is the symmetric difference operator.
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- The algorithm becomes:

**Algorithm 8.1:** Alternating Path Bipartite Matching

1. Let $A$ be an arbitrary (including empty) matching in $G = (V, F, E)$;
2. while There exists an augmenting path $S$ in $G$ do
   3. \[ A \leftarrow A \oplus S \]
Bipartite Matching

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   - $S \leftarrow A \oplus S$

- This can be made to run in $O(m^2n)$, where $|V| = m$, $|F| = n$, $m \leq n$. 
Bipartite Matching Example

Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.

$A = \emptyset$
Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.

\[ A = \{e_1, e_2, e_3\} \]
Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.

\[ \{ (2,7) \} \cup \{ (5,8) \} = \{ (2,7), (5,8) \} = A \]
Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.
Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).

\[
\{ (2,1), (5,8), (9,10) \} \oplus \{ (5,8), (3,1), (7,1) \} = \{ (2,1), (5,8), (9,10) \} \]

Prof. Jeff Bilmes
Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible). At this point, matching is maximum cardinality.
Matroid Intersection - recall from Lecture 5

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$. 
Matroid Intersection - recall from Lecture 5

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$. 

**Theorem 4.1**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$\min_{X \subseteq V} (r_1(X) + r_2(V \setminus X)) \quad (7)$$

In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} (r_1(X) + r_2(Y \setminus X)) \quad (8)$$
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Matroid Intersection - recall from Lecture 5

- Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$.
- While $(V, I_1 \cap I_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$.

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(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} \left( r_1(X) + r_2(Y \setminus X) \right)
$$
Let $V$ be our ground set.

Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into disjoint sets (disjoint union). Define a set of subsets of $V$ as

$$I = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \quad (9)$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.

Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$. 
Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
- Consider bipartite graph \( G = (V, F, E) \). Define two partition matroids \( M_V = (E, I_V) \), and \( M_F = (E, I_F) \).
Why might we want to do matroid intersection?

Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, I_V)$, and $M_F = (E, I_F)$.

$I \in I_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$ and $I \in I_F$ if $|I \cap (v, F)| \leq 1$ for all $v \in V$.
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Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
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- Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.

- For the bipartite graph case, therefore, this can be solved in polynomial time.
Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on the same underlying edges.
Matroid Intersection and Network Communication

- Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on the same underlying edges.
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Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on the same underlying edges.

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We may wish to find the maximum edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either $M_1$, $M_2$, or both).
Let \( G_1 = (V_1, E) \) and \( G_2 = (V_2, E) \) be two graphs on the same underlying edges.

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This is again a matroid intersection problem.
Matroid Intersection and TSP

- A **Hamiltonian cycle** is a cycle that passes through each node exactly once.
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From $G$ with $n$ nodes, create $G'$ with $n + 1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to $v_1^+, v_1^-$, and have all outgoing edges from $v_1$ come instead from $v_1^+$ and all edges incoming to $v_1$ go instead to $v_1^-$. 
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Let $M_2$ be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$. Let $M_3$ be the partition matroid having as independent sets those that have no more than one edge entering any node — i.e., $I \in \mathcal{I}(M_3)$ if $|I \cap \delta^-(v)| \leq 1$ for all $v \in V(G')$.

Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1$, $M_2$, and $M_3$. 

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EE595A/Spr 2011/Submodular Functions – Lecture 8 - April 27th, 2011
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Matroid Intersection and TSP

- A Hamiltonian cycle is a cycle that passes through each node exactly once.
- Given graph $G$, we may wish to find such a Hamiltonian cycle.
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- Let $M_1$ be the cycle matroid on $G'$.
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- Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1$, $M_2$, and $M_3$. 
Since TSP is NP-complete, we obviously can’t solve matroid intersections of 3 more matroids, unless P=NP.
Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless $P=NP$.

But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...
Recall: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 4.2**

*Matroid (by circuits)* Let $E$ be a set and $\mathcal{C}$ be a collection of nonempty subsets of $E$, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. $(C1)$ $\mathcal{C}$ is the collection of circuits of a matroid;
2. $(C2)$ if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;
3. $(C3)$ if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing $y$;
Lemma 4.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in $M$.

Proof.

Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
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Proof.

- Suppose, to the contrary, that there are two distinct circuits \( C_1, C_2 \) such that \( C_1 \cup C_2 \subseteq I \cup \{e\} \).
- Then \( \not\exists \in C_1 \cap C_2 \), and by (C2), there is a circuit \( C_3 \) of \( M \) s.t. \( C_3 \subseteq (C_1 \cup C_2) \setminus \{\not\exists\} \subseteq I \).
**Lemma 4.3**

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- Suppose, to the contrary, that there are two distinct circuits \( C_1, C_2 \) such that \( C_1 \cup C_2 \subseteq I \cup \{e\} \).
- Then \( x \in C_1 \cap C_2 \), and by (C2), there is a circuit \( C_3 \) of \( M \) s.t. \( C_3 \subseteq (C_1 \cup C_2) \setminus \{x\} \subseteq I \).
- This contradicts the independence of \( I \).
Lemma 4.3

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- Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
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- This contradicts the independence of $I$.

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$. 

$C(I, e) \subseteq E$
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. 
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \text{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$. 
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- Consider two matroids $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ and start with any $I \in I_1 \cap I_2$.
- Consider some $\nu_1 \notin \text{span}_1(I)$, so that $I + \nu_1 \in I_1$.
- If $I + \nu_1 \in I_2$, then $\nu_1$ is “augmenting”, and we can augment $I$ to $I + \nu_1$ and still be independent in both $M_1$ and $M_2$. 
Matroid Intersection Algorithm Idea

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If $I + v_1 \in I_2$, then $v_1$ is “augmenting”, and we can augment $I$ to $I + v_1$ and still be independent in both $M_1$ and $M_2$.

If $I + v_1 \notin I_2$, then $\exists C_2(I, v_1)$ a circuit in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}(I + v_1 - v_2)$) is again independent in $M_2$. It is also independent in $M_1$. 
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- Next choose a $\nu_3 \in \text{span}_1(I) - \text{span}_1(I - \nu_2)$ to recover what was lost in $I \cup \{\nu_1\}$ when we removed $\nu_2$ from it.
Matroid Intersection Algorithm Idea

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- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$. 
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
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- If $I + v_1 \notin \mathcal{I}_2$, then $\exists C_2(I, v_1)$ a circuit in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}(I + v_1 - v_2)$) is again independent in $M_2$. It is also independent in $M_1$.
- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$. 

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Matroid Intersection Algorithm Idea

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.
- But $I + v_1 - v_2 + v_3$ might not be independent in $M_2$ again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.
Matroid Intersection Algorithm Idea

- Hopefully (eventually) we’ll find an odd length sequence $S = (v_1, v_2, \ldots, v_n)$ such that we will be independent in both $M_1$ and $M_2$ and thus be one greater in size than $I$. 

Matroid Intersection Algorithm Idea

- Hopefully (eventually) we’ll find an odd length sequence 
  \( S = (v_1, v_2, \ldots, v_n) \) such that we will be independent in both \( M_1 \) and \( M_2 \) and thus be one greater in size than \( I \).

- We then replace \( I \) with \( I \ominus S \) (quite analogous to the bipartite matching case), and start again.
Graphic Matroid Intersection Example

Consider the following two graph $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$. Any edge is independent in both (an augmenting “sequence”) since a single edge can’t create a circuit starting at $I = \emptyset$. We start with $e_4$.  

![Diagram of graphs $G_1$ and $G_2$ with edges labeled $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$.]
Adding edge $I \leftarrow I + e_4$ creates a circuit neither in $M_1$ nor $M_2$. We can add another single edge w/o creating a circuit in either matroid.
$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we’re still independent in $M_2$, but no further single edge additions possible w/o creating a circuit (why?).
$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we’re still independent in $M_2$, but no further single edge additions possible w/o creating a circuit (why?). We need a multi-edge augmenting sequence if it exists.
Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in $I$, green is new). We choose $e_2 \in E - \text{span}_1(I)$, but now $I + e_2$ is not independent in $M_2$. 
Graphic Matroid Intersection Example

So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add.
Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add. In this case, we not only have $\text{span}_1(I + e_2) = \text{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$. 

![Graphic Matroid Intersection Example](image)
Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$.  

$I \not\subseteq I \ominus \text{set}(S)$
At this point, are any further augmenting sequences possible? Exercise.
Let $I$ be an intersection of two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ (i.e., $I \in I_1 \cap I_2$).
Alternating and Augmenting Sequences

- Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).
- Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

\begin{align*}
1 & \quad I + e_1 \in \mathcal{I}_1 \\
2 & \quad \text{For all even } i, \quad \text{span}_2(I \setminus S_i) = \text{span}_2(I) \quad \text{which implies that} \quad I \setminus S_i \in \mathcal{I}_2 \\
3 & \quad \text{For all odd } i, \quad \text{span}_1(S \setminus S_i) = \text{span}_1(I + e_1), \quad \text{and therefore} \quad I \setminus S_i \in \mathcal{I}_1. \\
4 & \quad \text{Lastly, if also,} \quad |S| = s \quad \text{is odd, and} \quad I \setminus S \in \mathcal{I}_2, \quad \text{then} \quad S \quad \text{is called an augmenting sequence w.r.t.} \quad I. \\
\end{align*}
Let $I$ be an intersection of two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ (i.e., $I \in I_1 \cap I_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in I_1$
Alternating and Augmenting Sequences

- Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).
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  1. $I + e_1 \in \mathcal{I}_1$
  2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$. 

Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in \mathcal{I}_1$
2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.
3. For all odd $i$, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$. 

Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then $S$ is called an augmenting sequence w.r.t. $I$. 
Let $I$ be an intersection of two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ (i.e., $I \in I_1 \cap I_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in I_1$
2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in I_2$.
3. For all odd $i$, $\text{span}_1(S \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in I_1$.

Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in I_2$, then $S$ is called an augmenting sequence w.r.t. $I$.
Alternating and Augmenting Sequences

- If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$. 
Alternating and Augmenting Sequences

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Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, if there is an augmenting sequence, then the intersection is not maximum.
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- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, if there is an augmenting sequence, then the intersection is not maximum.

- We next wish to show that, if the intersection is maximum, then there is an augmenting sequence.
Border graphs

We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$. 

- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create ← edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \{e_i\}$.
  
  If $e_i / \in \text{span}_1(I)$, then $e_i$ has in-degree zero (a source).

- Right-going edges: For each $e_i \in \text{span}_2(I) \setminus I$, create → edge $(e_i, e_j) \in Z$ for any $e_j \in C_2(I, e_i) \{e_i\}$.
  
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Border graphs

- We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$.

- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $\leftarrow$ edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$.

- Right-going edges: For each $e_i \in \text{span}_2(I) \setminus I$, create $\rightarrow$ edge $(e_i, e_j) \in Z$ for any $e_j \in C_2(I, e_i) \setminus \{e_i\}$.
Border graphs

- We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$.

- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $\leftarrow$ edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$.

- If $e_i \notin \text{span}_1(I)$, then $e_i$ has in-degree zero (a source).
We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$.

- **Left-going edges:** For each $e_i \in \text{span}_1(I) \setminus I$, create $\leftarrow$ edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$.

- **Right-going edges:** For each $e_i \in \text{span}_2(I) \setminus I$, create $\rightarrow$ edge $(e_i, e_j) \in Z$ for any $e_j \in C_2(I, e_i) \setminus \{e_i\}$.
We construct an auxiliary directed bipartite graph (border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$.

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Right-going edges: For each $e_i \in \text{span}_2(I) \setminus I$, create $\rightarrow$ edge $(e_i, e_j) \in Z$ for any $e_j \in C_2(I, e_i) \setminus \{e_i\}$.

If $e_i \not\in \text{span}_2(I)$, then $e_i$ has out-degree zero (a sink).
\{e_2, e_7, e_8\} are sources and \{e_1, e_3, e_6\} are sinks.
\{e_2, e_7, e_8\} are sources and \{e_1, e_3, e_6\} are sinks.

Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all of which are dipaths in the Border graph.
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Augmenting sequences are \((e_2, e_4, e_1)\), \((e_2, e_4, e_3)\), and \((e_2, e_4, e_6)\), all of which are dipaths in the Border graph.

Are there others?
Lemma 4.4

If $S$ is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then $S$ is an augmenting sequence w.r.t. $I$. 
Lemma 4.4

If $S$ is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then $S$ is an augmenting sequence w.r.t. $I$.

Lemma 4.5

Let $I$ and $J$ be intersections such that $|I| + 1 = |J|$. Then there exists a source-sink path $S$ in $B(I)$ where $S \subseteq I \ominus J$. 
Theorem 4.6

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \Theta I_{p+1}$ w.r.t. $I_p$. 
Identifying Augmenting Sequences

**Theorem 4.6**

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \Theta I_{p+1}$ w.r.t. $I_p$.

**Theorem 4.7**

An intersection is of maximum cardinality iff it admits no augmenting sequence.
Identifying Augmenting Sequences

**Theorem 4.6**

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. $I_p$.

**Theorem 4.7**

An intersection is of maximum cardinality iff it admits no augmenting sequence.

**Theorem 4.8**

For any intersection $I$, there exists a maximum cardinality intersection $I^*$ such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.
Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$. 
Matroid Partition Problem

- Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$.
- We wish to, if possible, partition $E$ into $k$ blocks, $I_1, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$. 
Matroid Partition Problem

- Suppose \( M_i = (E, \mathcal{I}_i) \) is a matroid and that we have \( k \) of them on the same ground set \( E \).
- We wish to, if possible, partition \( E \) into \( k \) blocks, \( I_1, i \in \{1, 2, \ldots, k\} \) where \( I_i \in \mathcal{I}_i \).
- Moreover, we want partition to be lexicographically maximum, that is \( |I_1| \) is maximum, \( |I_2| \) is maximum given \( |I_1| \), and so on.
Matroid Partition Problem

**Theorem 5.1**

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i$, $i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

\[
|A| \leq \sum_{i=1}^{k} r_i(A) \quad (10)
\]

where $r_i$ is the rank function of $M_i$. 

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Prof. Jeff Bilmes

EE595A/Spr 2011/Submodular Functions – Lecture 8 - April 27th, 2011
Theorem 5.1

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \subseteq \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$

where $r_i$ is the rank function of $M_i$.

- Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

$$|A| \leq kr(A) \ \forall A \subseteq E$$
Matroid Partition Problem

Theorem 5.1

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

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where $r_i$ is the rank function of $M_i$.

- Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

$$|A| \leq kr(A) \quad \forall A \subseteq E \quad (11)$$

- But considering vector of all ones $1 \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k} 1(A) \leq r(A) \quad \forall A \subseteq E \quad (12)$$
Recall definition of matroid polytope

\[ P_r = \left\{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \]  (13)
Matroid Partition Problem

- Recall definition of matroid polytope
  \[ P_r = \left\{ y \in \mathbb{R}^E_+ : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \] (13)

- Then we see that this special case of the matroid partition problem is just testing if \( \frac{1}{k} \mathbf{1} \in P_r \), a problem of testing the membership in matroid polyhedra.
Sources for Today’s Lecture

- Cunningham, “Testing Membership in Matroid Polyhedra”, 1984