EE595A – Submodular functions, their optimization and applications – Spring 2011

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Lecture 7 – April 20th, 2011
Announcements

- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.
- Please write down your email on paper for me.
- Engineering Discovery Days http://www.engr.washington.edu/alumcomm/openhouse.html, so no class on Friday.
A polytope can be defined in a number of ways, two of which include

Theorem 2.1

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- $P$ is the convex hull of a finite set of points.
- If it is a bounded intersection of halfspaces, that is there exists matrix $A$ and vector $b$ such that

$$P = \{ x : Ax \leq b \}$$  \hspace{1cm} (1)
A polytope can be defined in a number of ways, two of which include:

**Theorem 2.1**

A subset \( P \subseteq \mathbb{R}^E \) is a polytope iff it can be described in either of the following (equivalent) ways:

- \( P \) is the convex hull of a finite set of points.
- If it is a **bounded** intersection of halfspaces, that is there exists matrix \( A \) and vector \( b \) such that

\[
P = \{ x : Ax \leq b \} \tag{1}
\]

This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.
For each \( I \in \mathcal{I} \) of a matroid \( M = (E, \mathcal{I}) \), we can form the incidence vector \( \mathbf{1}_I \).
Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is
  \[
  P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \mathbf{1}_I \right\}
  \] (2)
For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I$.

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P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} 1_I \right\}
\] (2)

Now take the rank function $r$ of $M$, and define the following polyhedron:
\[
P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\}
\] (3)
Independence Polyhedra

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Theorem 2.2

\[
P_r = P_{\text{ind. set}}
  \]  
  (4)
Matroid Polyhedron in 2D

\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq V \right\} \tag{5} \]

- Consider this in two dimensions. We have equations of the form:
  \[ x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{6} \]
  \[ x_1 \leq r(\{v_1\}) \tag{7} \]
  \[ x_2 \leq r(\{v_2\}) \tag{8} \]
  \[ x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{9} \]
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  \[ x_2 \leq r(\{v_2\}) \quad (8) \]
  \[ x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (9) \]

- Because \( r \) is submodular, we have
  \[ r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (10) \]
  so since \( r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\}) \), the last inequality is either touching or active.
And, if \( v_2 \) is a loop ...

\[
\begin{align*}
x_2 & \leq r(\{v_2\}) \\
x_2 & \geq 0 \\
x_1 & \geq 0 \\
x_1 & \leq r(\{v_1\})
\end{align*}
\]
Matroid Polyhedron in 2D

\[ x_2 \leq r(\{v_2\}) \]
\[ x_2 \geq 0 \]
\[ x_1 \geq 0 \]
\[ x_1 \leq r(\{v_1\}) \]
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \]

Possible
Not Possible
Not
Possible

Box polytope
Matroid Polyhedron in 3D

\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \quad (11) \]

- Consider this in three dimensions. We have equations of the form:
  - \[ x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \quad (12) \]
  - \[ x_1 \leq r(\{v_1\}) \quad (13) \]
  - \[ x_2 \leq r(\{v_2\}) \quad (14) \]
  - \[ x_3 \leq r(\{v_3\}) \quad (15) \]
  - \[ x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (16) \]
  - \[ x_2 + x_3 \leq r(\{v_2, v_3\}) \quad (17) \]
  - \[ x_1 + x_3 \leq r(\{v_1, v_3\}) \quad (18) \]
  - \[ x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \quad (19) \]
Consider the simple cycle matroid on a graph consisting of a 3-cycle, \( G = (V, E) \) with matroid \( M = (E, I) \) where \( I \in \mathcal{I} \) is a forest.
Consider the simple cycle matroid on a graph consisting of a 3-cycle, \( G = (V, E) \) with matroid \( M = (E, \mathcal{I}) \) where \( I \in \mathcal{I} \) is a forest.

So any set of either one or two edges is independent, and has rank equal to cardinality.
Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.

So any set of either one or two edges is independent, and has rank equal to cardinality.

The set of three edges is dependent, and has rank 2.
Matroid Polyhedron in 3D

Two view of $P_r$ associated with a matroid $M = (E, I) = (\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\})$.
Matroid Polyhedron in 3D

$P_r$ associated with the “free” matroid in 3D.
Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
Greedy solves a linear programming problem

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- In fact, we see that the LP problem with exponential number of constraints $\max \{w^T x : x \in P_{\text{ind. set}}\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:
Greedy solves a linear programming problem

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- In fact, we see that the LP problem with exponential number of constraints \( \max \{ w^T x : x \in P_{\text{ind. set}} \} \) is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

**Theorem 2.3**

The LP problem \( \max \{ w^T x : x \in P_{\text{ind. set}} \} \) can be solved exactly using the greedy algorithm.

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.
Consider convex hull of indicator vectors of bases of a matroid. By the same argument, this will be the same as the following polytope

\begin{align*}
x & \geq 0 \\
x(A) & \leq r(A) \quad \forall A \subseteq V \\
x(V) & = r(V)
\end{align*}
Base Polytope Equivalence

- Consider convex hull of indicator vectors of bases of a matroid. By the same argument, this will be the same as the following polytope:

\[
\begin{align*}
\rho_c & \quad x \geq 0 \\
\bigwedge & \quad x(A) \leq r(A) \quad \forall A \subseteq V \\
\bigwedge & \quad x(V) = r(V)
\end{align*}
\] (20-22)

- By essentially the same argument as above, we have shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 20-22 above.
Spanning set polytope

- Consider convex hull of incidence vectors of spanning sets of a matroid $M$, and call this $P_{\text{spanning}}(M)$. 
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**Theorem 3.1**

The spanning set polytope is determined by the following equations:

$$0 \leq x_e \leq 1 \quad \text{for } e \in E$$

$$x(A) \geq r(E) - r(E \setminus A) \quad \text{for } A \subseteq E$$
Proof.

- Recall that any $A$ is spanning in $M$ iff $E \setminus A$ is independent in $M^*$ (the dual matroid).
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- Recall that any $A$ is spanning in $M$ iff $E \setminus A$ is independent in $M^*$ (the dual matroid).
- Therefore, for any $x \in \mathbb{R}^E$, we have that

$$x \in P_{\text{spanning}}(M) \iff 1 - x \in P_{\text{ind. set}}(M^*) \quad (25)$$
Spanning set polytope

Proof.

This follows since if \( x \in P_{\text{spanning}}(M) \), we can represent \( x \) as a convex combination:

\[
x = \sum_{i} \lambda_i 1_{A_i}
\]

where \( A_i \) is spanning in \( M \).
Spanning set polytope

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  \[
  x = \sum_{i} \lambda_i 1_{A_i}
  \]  
  (26)
  
  where \( A_i \) is spanning in \( M \).

- Consider
  \[
  1_E - x = \sum_{i} \lambda_i 1_{E \setminus A_i}
  \]  
  (27)

  so \( 1_E - x \) is a convex combination of independent sets in \( M^* \) and so \( 1_E - x \in P_{\text{ind. set}}(M^*) \).

...
Spanning set polytope

Proof.

which means that

\[ 1 \mathbf{1}_E - \mathbf{x} \geq 0 \tag{28} \]

\[ 1_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \tag{29} \]

And we know the dual rank function is

\[ r_{M^*}(A) = |A| + r_{M}(E \setminus A) - r_{M}(E) \]

...
Spanning set polytope

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- which means that
  \[ 1 - x \geq 0 \]  \hspace{1cm} (28)
  \[ 1_A - x(A) = |A| - x(A) \leq r^{\star}_M(A) \text{ for } A \subseteq E \] \hspace{1cm} (29)

And we know the dual rank function is

\[ r^{\star}_M(A) = |A| + r_M(E \setminus A) - r_M(E) \] \hspace{1cm} (29.5)

- giving
  \[ x(A) \geq r_M(A) - r_M(E \setminus A) \text{ for all } A \subseteq E \] \hspace{1cm} (30)
Matroids
where are we going with this?

- We’ve been discussing results about matroids (independence polytope, etc.).
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By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it is worth showing a result for a general submodular function first.
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- We’ve been discussing results about matroids (independence polytope, etc.).
- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it is worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...
A vector form of rank

- Recall the definition of rank from a matroid $M = (E, I)$.
  \[
  \text{rank}(A) = \max \{ |I| : I \subseteq A, I \in I \} \quad (31)
  \]

Given some polytope, say $P$, we can define a form of “vector rank” relative to this polytope in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank as:

\[
\text{rank}(x) = \max \{ y(E) : y \leq x, y \in P \} \quad (32)
\]

In general, this might be hard to compute and/or have ill-defined properties. We next look at an object that puts constraints based on this form of rank.
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\[ y(E) = \sum_{e \in E} g_e = 1^T \cdot y \]
Polymatroidal polyhedron (or a “polymatroid”)

Definition 4.1 (polymatroid)

A polymatroid is a compact set \( P \subseteq \mathbb{R}_+^E \) satisfying

1. \( 0 \in P \)
2. If \( y \leq x \in P \) then \( y \in P \) (called down monotone).
3. For any \( x \in \mathbb{R}_+^E \), any maximal vector \( y \in P \) with \( y \leq x \) (called a \( P \)-basis of \( x \)), has the same component sum \( y(E) \). That is for any \( y^1, y^2 \in \arg\max \{ y(E) : y \leq x, y \in P \} \) we have \( y^1(E) = y^2(E) \).
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Another way of saying this is that a polymatroid is a compact set that is zero containing, down monotone, and any maximal vector $y$ in $P$, bounded by another vector $x$, has the same vector rank. Compare with the definition of a matroid: a set system that is empty-set containing, down closed, and any maximal set $I$ in $I$, bounded by another set $A$, has the same vector rank.
A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

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Polymatroidal polyhedron (or a "polymatroid")

Left: \( \exists \) multiple maximal \( y \leq x \) Right: \( \exists \) only one maximal \( y \leq x \),

- On the left, we see there are multiple possible maximal such \( y \in P \) that are \( y \leq x \). Each such \( y \) must have the same value \( y(E) \).
Polymatroidal polyhedron (or a “polymatroid”)

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- On the left, we see there are multiple possible maximal such \( y \in P \) that are \( y \leq x \). Each such \( y \) must have the same value \( y(E) \).
- On the right, there is only one maximal \( y \in P \). Since there is only one, the \( \forall y \) condition is vacuous.
Polymatroidal polyhedron (or a “polymatroid”)

\[ \exists \text{ only one maximal } y \leq x . \]

- If \( x \in P \) already, then \( x \) is its own \( P \)-basis, i.e., it is a self \( P \)-basis.
Polymatroidal polyhedron (or a “polymatroid”)

∃ only one maximal $y \leq x$.

- If $x \in P$ already, then $x$ is its own $P$-basis, i.e., it is a **self $P$-basis**.
- In a matroid, a base of $A$ is the maximally contained independent set. If $A$ is already independent, then $A$ is a self-base of $A$ (as we saw in lecture 3).
Polymatroid as well??

Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such $y$ must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2$, we see this is not a polymatroid.
Polymatroid as well??

Left and right: \( \exists \) multiple maximal \( y \leq x \) as indicated.

- On the left, we see there are multiple possible maximal such \( y \in P \) that are \( y \leq x \). Each such \( y \) must have the same value \( y(E) \), but since the equation for the curve is \( y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2 \), we see this is not a polymatroid.

- On the right, we have a similar situation, just the set of potential values that must have the \( y(E) \) condition changes, but the values of course are still not constant.
Polymatroidal polyhedron (or a “polymatroid”)

Note that if $x$ contains any zeros (i.e., suppose that $x \in \mathbb{R}^E_+$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. 
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- Therefore, in this case, it is the non-zero elements of $x$, corresponding to elements $S$, determine the common component sum.
Polymatroidal polyhedron (or a “polymatroid”)

- Note that if \( x \) contains any zeros (i.e., suppose that \( x \in \mathbb{R}^E_+ \) has \( E \setminus S \) s.t. \( x(E \setminus S) = 0 \)), then this also forces \( y(E \setminus S) = 0 \), so that \( y(E) = y(S) \).
- Therefore, in this case, it is the non-zero elements of \( x \), corresponding to elements \( S \), determine the common component sum.
- We might give a “name” to this component sum, say \( f(S) \) for any given set \( S \) of non-zero elements of \( x \).

\[
\begin{align*}
\text{possible } y &= f(1) \\
\end{align*}
\]
A polymatroid function is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P_f$ associated with a polymatroid function as follows

$$ P_f = \left\{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \right\} $$ (33)

$$ = \left\{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \right\} $$ (34)

$f$ is submodular. $\omega : E \rightarrow \mathbb{R}$, $\omega(e) = f(E) - f(E \setminus \{e\}) \leq f(A) - f(A \setminus \{e\})$

$\overline{f} = f - \omega - f(b)$, $f(A) = f(A) - \omega(A) - f(\emptyset)$
Associated polyhedron with a polymatroid function

\[ P_f = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \right\} \] (35)

- Consider this in three dimensions. We have equations of the form:
  \[ x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \] (36)
  \[ x_1 \leq f(\{v_1\}) \] (37)
  \[ x_2 \leq f(\{v_2\}) \] (38)
  \[ x_3 \leq f(\{v_3\}) \] (39)
  \[ x_1 + x_2 \leq f(\{v_1, v_2\}) \] (40)
  \[ x_2 + x_3 \leq f(\{v_2, v_3\}) \] (41)
  \[ x_1 + x_3 \leq f(\{v_1, v_3\}) \] (42)
  \[ x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \] (43)
Associated polyhedron with a polymatroid function

- Consider concave function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Then $f(S) = g(|S|)$ is submodular.
Consider concave function on integers: \( g(0) = 0, g(1) = 3, g(2) = 4, \) and \( g(3) = 5.5 \). Then \( f(S) = g(|S|) \) is submodular.

Observe: \( P_f \) (at two views):
Consider the graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S)$ is count of any edges within $S$ or between $S$ and $V \setminus S$, so that $f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.
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Consider the graph cut function on the simple chain graph \( v_1 - v_2 - v_3 \). That is, \( f(S) \) is count of any edges within \( S \) or between \( S \) and \( V \setminus S \), so that \( f(S) + f(V \setminus S) - f(V) \) is the standard graph cut.

Observe: \( P_f \) (at two views):

\[
\begin{align*}
V_1 & \quad V_2 \\
0 & \quad 0.5 & \quad 1 \\
0.5 & \quad 1 & \quad 1.5 & \quad 2
\end{align*}
\]

which axis is which?
Associated polyhedron with a polymatroid function

Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$. 
Associated polyhedron with a polymatroid function

Consider: \( f(\emptyset) = 0, \ f(\{v_1\}) = 1.5, \ f(\{v_2\}) = 2, \ f(\{v_1, v_2\}) = 2.5, \ f(\{v_3\}) = 3, \ f(\{v_3, v_1\}) = 3.5, \ f(\{v_3, v_2\}) = 4, \ f(\{v_3, v_2, v_1\}) = 4.3. \)

Observe: \( P_f \) (at two views):
Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

Observe: $P_f$ (at two views):

- Which axis is which?
Associated polyhedron with a polymatroid function

- Consider modular function $w : V \rightarrow \mathbb{R}^+$ as $w = (1, 1.5, 2)^T$, and then the submodular function $f(S) = \sqrt{w(S)}$. 
Consider modular function \( w : V \rightarrow \mathbb{R}_+ \) as \( w = (1, 1.5, 2)^T \), and then the submodular function \( f(S) = \sqrt{w(S)} \).

Observe: \( P_f \) (at two views):
Consider modular function \( w : V \rightarrow \mathbb{R}_+ \) as \( w = (1, 1.5, 2)^\top \), and then the submodular function \( f(S) = \sqrt{w(S)} \).

Observe: \( P_f \) (at two views):

- \( \vee_1 \)
- \( \vee_2 \)

which axis is which?
A polymatroid function’s polyhedron is a polymatroid.

\[
f(A) = \max \left( y(E) : y(E) \leq 1, y \in P_f \right)
\]

\[
\chi(E \setminus A) + f(A) : E \setminus A \leq E
\]

**Theorem 4.3**

Let \( f \) be a polymatroid function defined on subsets of \( E \). For any \( x \in \mathbb{R}_+^E \), and any \( P_f \)-basis \( y^x \) of \( x \), we have that the component sum of \( y \) is

\[
\max (y(E) : y \leq x, y \in P_f) = y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E)
\]

As a consequence, \( P_f \) is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f$ since $f$ is non-negative.
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f$ since $f$ is non-negative.
- Also, if any $y \in P_f$ then any $x \leq y$ is also such that $x \in P_f$. So, $P_f$ is down-monotone, and we’ve got two of the required properties for a polymatroid.
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Proof.

- Clearly $0 \in P_f$ since $f$ is non-negative.
- Also, if any $y \in P_f$ then any $x \leq y$ is also such that $x \in P_f$. So, $P_f$ is down-monotone, and we’ve got two of the required properties for a polymatroid.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y \in P_f$ with $y \leq x$ (i.e., a $P_f$-basis $y$ of $x$).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f$ since $f$ is non-negative.
- Also, if any $y \in P_f$ then any $x \leq y$ is also such that $x \in P_f$. So, $P_f$ is down-monotone, and we’ve got two of the required properties for a polymatroid.
- Now suppose that we are given an $x \in \mathbb{R}^E_+$, and maximal $y \in P_f$ with $y \leq x$ (i.e., a $P_f$-basis $y$ of $x$).
- Now, because $y \in P_f$, we have that $y(A) \leq f(A)$ for all $A$. 

A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Our goal is to show that such a $y$ is such that $y(E) = \text{const}$ that is dependent only on $x$ (and in this case $f$ since that defines the polytope).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Our goal is to show that such a \( y \) is such that \( y(E) = \text{const} \) that is dependent only on \( x \) (and in this case \( f \) since that defines the polytope).
- Doing so will thus establish that \( P_f \) is a polymatroid
A polymatroid function’s polyhedron is a polymatroid.

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- Our goal is to show that such a $y$ is such that $y(E) = \text{const}$ that is dependent only on $x$ (and in this case $f$ since that defines the polytope).
- Doing so will thus establish that $P_f$ is a polymatroid.
- We show that this constant is given by
  \[
y(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \tag{45}
  \]
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Our goal is to show that such a $y$ is such that $y(E) = \text{const}$ that is dependent only on $x$ (and in this case $f$ since that defines the polytope).

- Doing so will thus establish that $P_f$ is a polymatroid.

- We show that this constant is given by

$$y(E) = \min \left( x(A) + f(E \setminus A) : A \subseteq E \right) \quad (45)$$

- Now, for any $P_f$-basis $y$ of $x$, and any $A \subseteq E$, we have that

$$y(E) = y(A) + y(E \setminus A) \quad (46)$$

$$\leq x(A) + f(E \setminus A) \quad (47)$$

*This follows since $y \leq x$ and since $y \in P_f$.***
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Our goal is to show that such a $y$ is such that $y(E) = \text{const}$ that is dependent only on $x$ (and in this case $f$ since that defines the polytope).

- Doing so will thus establish that $P_f$ is a polymatroid.

- We show that this constant is given by

$$y(E) = \min (x(A) + f(E \setminus A) : A \subseteq E)$$  \hspace{1cm} (45)

- Now, for any $P_f$-basis $y$ of $x$, and any $A \subseteq E$, we have that

$$y(E) = y(A) + y(E \setminus A) \leq x(A) + f(E \setminus A).$$  \hspace{1cm} (46)

This follows since $y \leq x$ and since $y \in P_f$.

- So if we can find one such $A$ where equality holds, the above min result follows.

...
A polymatroid function’s polyhedron is a polymatroid.

Proof.

Now fix $y$ and call a set $B \subseteq E$ **tight** if $y(B) = f(B)$. The union of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C)$$  \hspace{1cm} (48)

$$= y(B \cap C) + y(B \cup C)$$  \hspace{1cm} (49)

$$= f(B \cap C) + f(B \cup C)$$  \hspace{1cm} (50)

$$\leq f(B) + f(C)$$  \hspace{1cm} (51)

which requires equality everywhere (i.e., $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, both also tight).

...
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Now fix \( y \) and call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union of tight sets \( B, C \) is again tight, since

\[
    f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C) \leq f(B) + f(C)
\]

which requires equality everywhere (i.e., \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), both also tight).

- Given a \( j \in E \), either \( y_j \) is saturated due to \( x \) (so \( y_j = x_j \)) or \( j \) is an element of some tight set. Let \( E \setminus A \) be the union of all such tight sets, so we have

\[
y(E) = y(A) + y(E \setminus A) = x(A) + f(E \setminus A)
\] as desired.
We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\), but we now see that they are equivalent.
Matroid case is now a special case, where we have that:

**Corollary 5.1**

*We have that:*

\[
\max \{ y(A) : y \in P_{\text{ind. set}}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\]

(53)

where \( r_M \) is the matroid rank function of some matroid.
Most violated inequality problem

- Consider

\[ P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \quad (54) \]
Most violated inequality problem

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Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \not\in P_r \).

The most violated inequality when \( x \) is considered w.r.t. \( P_r \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e.,

\[ \max \{ x(A) - r_M(A) : A \subseteq E \} \]
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- This corresponds to \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \).
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- More importantly, \( \min \left\{ r_M(A) + x(E \setminus A) : A \subseteq E \right\} \) a form of submodular function minimization, namely
  \[ \min \left\{ r_M(A) - x(A) : A \subseteq E \right\} \] for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).
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- The most violated inequality when \( x \) is considered w.r.t. \( P_r \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e.,
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  \[ \min \{ r_M(A) - x(A) : A \subseteq E \} \] for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

- We study this case first ...
In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}^E_+$;
In particular, we will solve the following problem:

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- **find**: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $x(A) = r_M(A) + x(E \setminus A)$. 

By the above theorem, the existence of such an $A$ will certify that $x(E)$ is maximal in $P_{\text{ind. set}}$. This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$). This will also run in polynomial time.
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Approach

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- find: a maximizing \( y \in P_{\text{ind. set}} \) with \( y \leq x \), and moreover (as a byproduct of the algorithm), express \( y \) as a convex combination of incidence vectors of independent sets in \( M \), and also return a set \( A \subseteq E \) that satisfies \( x(A) = r_M(A) + x(E \setminus A) \).
- By the above theorem, the existence of such an \( A \) will certify that \( x(E) \) is maximal in \( P_{\text{ind. set}} \).
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- By the above theorem, the existence of such an $A$ will certify that $x(E)$ is maximal in $P_{\text{ind. set}}$.
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$).
- This will also run in polynomial time.
Recall: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 5.2

Matroid (by circuits) Let $E$ be a set and $\mathcal{C}$ be a collection of nonempty subsets of $E$, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. (C1) $\mathcal{C}$ is the collection of circuits of a matroid;
2. (C2) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;
3. (C3) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing $y$;
Lemma 5.3

Let \( I \in \mathcal{I}(M) \), and \( e \in E \), then \( I \cup \{e\} \) contains at most one circuit in \( M \).

Proof.

Suppose, to the contrary, that there are two distinct circuits \( C_1, C_2 \) such that \( C_1 \cup C_2 \subseteq I \cup \{e\} \).

In general, let \( C(I, e) \) be the unique circuit associated with \( I \cup \{e\} \).
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Then \( x \in C_1 \cap C_2 \), and by (C2), there is a circuit \( C_3 \) of \( M \) s.t. \( C_3 \subseteq (C_1 \cup C_2) \setminus \{x\} \subseteq I \).

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- This contradicts the independence of \( I \).

In general, let \( C(I, e) \) be the unique circuit associated with \( I \cup \{e\} \).
Idea of the algorithm

- We build up $y$ from the ground up.
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- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{I_i}$. 
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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we’ll keep $y \leq x$. 
Define associated digraph $G$ as follows.

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   - Make a directed edge $(s, e)$ for all $e \in E$ such that $y(e) < x(e)$.
   - Make a directed edge $(e, f)$ for every pair of distinct elements $e, f \in E$ such that for some $i \in J$, $f \in C(I_i, e)$.
   - Make a directed edge $(e, t)$ for $e \in E$ such that for some $i \in J$, $e / \in I_i \cup \{e\} \in I(i.e., all $e$ that can be added to some $I_i$ and keep it independent).

Algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1.$
Associated graph

- Define associated digraph $G$ as follows.
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- Make a directed edge $(e, t)$ for $e \in E$ such that for some $i \in J$, $e \not\in I_i$ such that $I_i \cup \{e\} \in I$ (i.e., all $e$ that can be added to some $I_i$ and keep it independent).
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Make a directed edge $(e, t)$ for $e \in E$ such that for some $i \in J$, $e \notin I_i$, we have $I_i \cup \{e\} \in I$ (i.e., all $e$ that can be added to some $I_i$ and keep it independent).
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Make a directed edge $(e, f)$ for every pair of distinct elements $e, f \in E$ such that for some $i \in J$, $f \in C(l_i, e)$.

Make a directed edge $(e, t)$ for $e \in E$ such that for some $i \in J$, $e \not\in l_i$, we have $l_i \cup \{e\} \in \mathcal{I}$ (i.e., all $e$ that can be added to some $l_i$ and keep it independent).

Algorithm starts with $y = 0$, $J = \{0\}$, $l_0 = \emptyset$, and $\lambda_0 = 1$. 

Prof. Jeff Bilmes
EE595A/Spr 2011/Submodular Functions – Lecture 7 - April 20th, 2011
Augmenting path theorem

**Theorem 5.4**

*If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) \geq y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).*
Sources for Today’s Lecture

Cunningham-1984, Schrijver-2003, Welsh-1973,