EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ee.washington.edu/class/235/2011wtr/index.html

Lecture 5 - April 13th, 2011

Announcements

- HW1 is on the web page now, at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/hw1.pdf
- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.

Matroid

Definition 2.1 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed)

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Matroids

In fact, we can use the rank of a matroid for its definition.

Theorem 2.2 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *forall* $A, B \subseteq E$ *(submodular)*

- So submodular non-negative integral monotone non-decreasing cardinality bounded is necessary and sufficient to define the matroid.

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i).
- A family $(v_i : i \in I)$ for index set I is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct.

Definition 2.3 (transversal)

Given a set system $(V, \mathcal{V})$, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{1}$$

-
- Note that due to it being a bijection, all of I and T are "covered" (so this makes things distinct).

Transversal Matroid Rank

- Transversal matroid has rank
$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{2}$$
- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general?

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- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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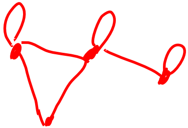
- There is no reason in a matroid such an A could not consist of a single element.
- Such an $\{a\}$ is called a loop.
- In a linear matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1.



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- Such an $\{a\}$ is called a loop.
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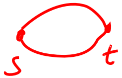
- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.



Representable

Definition 3.1

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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Definition 3.2

linear matroids on a field Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 3.3

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

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Representability of Transversal Matroids

- Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
- In particular:

Theorem 3.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 3.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Dual of a Matroid

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- Recall, in cycle matroid of a graph, a spanning set of G is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).

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- Recall, in cycle matroid of a graph, a spanning set of G is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
- Since the smallest spanning sets are bases, we see that the bases of M are complements of the bases of M^*.

Dual of a Matroid

Theorem 4.1

Let M^* be defined as on previous slide. Then M^* is a matroid.

Proof.

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- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.

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Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.

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- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of M. Also, $V \setminus I$ contains a base of M.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can find a base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

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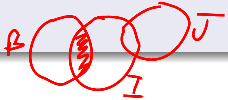
- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can find a base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since B and J are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, B' and I are disjoint.



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- Now $J \setminus I \not\subseteq B'$, since otherwise: (i.e., assuming $J \setminus I \subseteq B'$)

$$|B| = |B \cap I| + |B \setminus I| \tag{4}$$
$$\leq |I \setminus J| + |B \setminus I| \tag{5}$$
$$< |J \setminus I| + |B \setminus I| \leq |B'| \tag{6}$$

which is a contradiction.

Since $J \setminus I \subseteq B'$, $B \setminus I \subseteq B'$ ⇒ $J \setminus I \cup B \setminus I \subseteq B'$ & J & B disjoint ⇒ $|J \setminus I| + |B \setminus I| \leq |B'|$

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- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.

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  which is a contradiction.
- Therefore, $J \setminus I \subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So B' is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M, and therefore $I \cup \{v\} \in \mathcal{I}^*$.

□

Dual Matroid Rank

Theorem 4.2

The rank function r_{M^*} of the dual matroid M^* may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*

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- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

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or

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

Example duality: cocycle matroid

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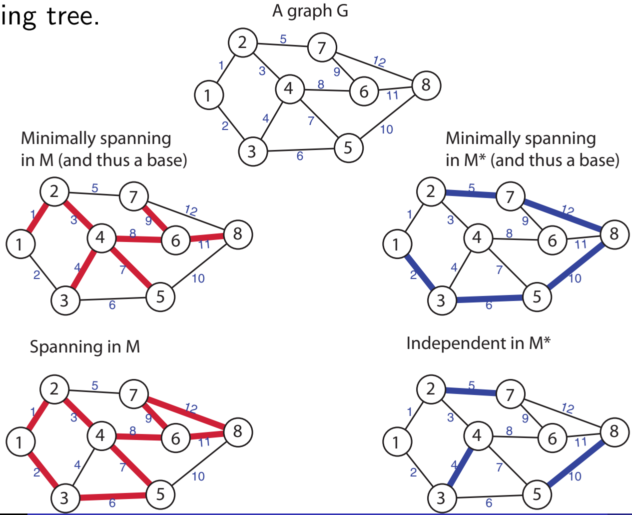

Matroid and the greedy algorithm

- Let $\mathcal{I}$ be a set of subsets of E that is down-closed. Consider a non-negative modular weight function $w : E \to \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

Theorem 5.1

Let $\mathcal{I}$ be a non-empty collection of subsets of a set E, down-closed (i.e., an independence system). Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Recall: Matroids by bases

Theorem 5.2

Matroid (by bases) Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."

Matroid and the greedy algorithm

proof of Theorem 5.1.

- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.

...

Matroid and the greedy algorithm

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- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \ldots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).

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- A is a base of M, and let $B = (b_1, \ldots, b_r)$ be any another base of M with elements also ordered decreasing by weight.

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- A is a base of M, and let $B = (b_1, \ldots, b_r)$ be any another base of M with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

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Matroid and the greedy algorithm

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.
- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

□

Matroid and the greedy algorithm

converse proof of Theorem 5.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We'll show $(E, \mathcal{I})$ is a matroid.
- Down monotonicity already holds (since we've started with an independence system).

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Matroid and the greedy algorithm

converse proof of Theorem 5.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We'll show $(E, \mathcal{I})$ is a matroid.
- Down monotonicity already holds (since we've started with an independence system).
- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on V, and define $k = |I|$.

$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in S \setminus (I \cup J) \end{cases} \tag{10}$$

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- Therefore, $(E, \mathcal{I})$ must be a matroid.

□

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then
$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{12}$$
is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid.
$r(M_Y) = r(Y)$

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is considered a deletion of X from M, and is often written $M \setminus Z$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$.

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$$r_{M/Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X|Z) \tag{14}$$

(see equations 57-60 from lecture 2).

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  (see equations 57-60 from lecture 2).
- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$ $(V, \mathcal{I})$ a matroid?

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Matroid Intersection

max size common independent set. $X \in \mathcal{I}_1$, $X \in \mathcal{I}_2$

find max $|X|$ s.t. $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

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Theorem 6.1

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. then the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$r_1 * r_2 = \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{15}$$

$$r_1 * r_2(Y) = \min_{X \subseteq Y} \left[r_1(X) + r_2(Y \setminus X) \right]$$

Matroid Union

Definition 6.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as $M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \cup V_2 \cup \dots \cup V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k = \{I_1 \cup I_2 \cup \dots \cup I_k | I_1 \in \mathcal{I}_1, \dots, I_k \in \mathcal{I}_k\} \tag{16}$$

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Theorem 6.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \dots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left(|Y \setminus X| + r_1(X \cap V_1) + \dots + r_k(X \cap V_k) \right) \tag{17}$$

for any $Y \subseteq V_1 \cup \dots V_k$.

Scratch Paper

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Sources for Today's Lecture

Korte,Vygen-2005, Vondrak-2010, Schrijver-2003, Oxley-1992, Welsh-1973, Goemans-2010, Piff,Welsh-1970.