Announcements

- HW1 is on the web page now, at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/hw1.pdf
- It is due, Tuesday, April 26th, 11:45pm
- All submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.
Definition 2.1 (Matroid)

A set system \((E, \mathcal{I})\) is a **Matroid** if

(I1') \(\emptyset \in \mathcal{I}\)

(I2') \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (down-closed)

(I3') \(\forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}\)
In fact, we can use the rank of a matroid for its definition.

**Theorem 2.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

1. (R1) $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
2. (R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. (R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ forall $A, B \subseteq E$ (submodular)

So submodular non-negative integral monotone non-decreasing cardinality bounded is necessary and sufficient to define the matroid.
System of Distinct Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)).
- A family \((v_i : i \in I)\) for index set \(I\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
- In a system of distinct representatives, there is a requirement for the representatives to be distinct.

Definition 2.3 (transversal)

Given a set system \((V, \mathcal{V})\), a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[ x \in V_{\pi(x)} \text{ for all } x \in T \]  \hspace{1cm} (1)

- Note that due to it being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct).
Transversal Matroid Rank

- Transversal matroid has rank
  \[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  

- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general?
Transversal Matroid Rank

Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \] (2)

Therefore, this function is submodular.

Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:
A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
- There is no reason in a matroid such an $A$ could not consist of a single element.
A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

There is no reason in a matroid such an $A$ could not consist of a single element.

Such an $\{a\}$ is called a loop.
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is **circuit** if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

- There is no reason in a matroid such an $A$ could not consist of a single element.

- Such an $\{a\}$ is called a **loop**.

- In a matric (i.e., linear) matroid, the only such loop is the value $0$, as all non-zero vectors have rank 1.
Matroid loops

- A circuit in a matroids is well defined, a subset \( A \subseteq E \) is circuit if it is an inclusionwise minimally dependent set (i.e., if \( r(A) < |A| \) and for any \( a \in A \), \( r(A \setminus \{a\}) = |A| - 1 \)).
- There is no reason in a matroid such an \( A \) could not consist of a single element.
- Such an \( \{a\} \) is called a loop.
- In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1.
- Note, we also say that two elements \( s, t \) are said to be parallel if \( \{s, t\} \) is a circuit.
Definition 3.1

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).
Representable

Definition 3.1

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi: V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
Definition 3.1

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
- We can more generally define matroids on a field.
Representable

Definition 3.1

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as $GF(p)$ where $p$ is prime (such as $GF(2)$).
- We can more generally define matroids on a field.

Definition 3.2

Linear matroids on a field Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $X_{ij} \in F$ for some field, and let $I$ be the set of subsets of $E$ such that the columns of $X$ are linearly independent over $F$. 
Definition 3.1

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
- We can more generally define matroids on a field.

Definition 3.3

Any matroid isomorphic to a linear matroid on a field is called **representable over $\mathbb{F}$**.
Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.

In particular:

**Theorem 3.4**

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.
Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 3.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$. It can be shown that this is a matroid and is representable. However, this matroid is not isomorphic to any transversal matroid.
Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 3.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

- It can be shown that this is a matroid and is representable.
Converse: Representability of Transversal Matroids

The converse is not true, however.

**Example 3.5**

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$.

- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.
Given a matroid \( M = (V, \mathcal{I}) \), a dual matroid \( M^* \) can be defined in a way such that \((M^*)^* = M\).
Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.

We define a set

$$\mathcal{I}^* = \{ I \subseteq V : V \setminus I \text{ is a spanning set of } M \}$$

(3)
Given a matroid $M = (V, I)$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.

We define a set

$$ \mathcal{I}^* = \{ I \subseteq V : V \setminus I \text{ is a spanning set of } M \} \quad (3) $$

Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.

We define a set
\[
\mathcal{I}^* = \{ I \subseteq V : V \setminus I \text{ is a spanning set of } M \}\] (3)

Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).

Since the smallest spanning sets are bases, we see that the bases of $M$ are complements of the bases of $M^*$.  

Dual of a Matroid

**Theorem 4.1**

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

**Proof.**

- Clearly $\emptyset \in I^*$, so (I1') holds.
**Theorem 4.1**

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

**Proof.**

- Clearly $\emptyset \in I^*$, so (I1') holds.

- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in $M$, so must $V \setminus I$. Therefore, (I2') holds.
Theorem 4.1

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. 

...
**Theorem 4.1**

*Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.*

**Proof.**

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$. 

...
Dual of a Matroid

Theorem 4.1

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Consider $I, J \in \mathcal{I}$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$.
- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can find a base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

...
**Theorem 4.1**

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

**Proof.**

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$.
- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can find a base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since $B$ and $J$ are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, $B'$ and $I$ are disjoint.

...
Dual of a Matroid

Theorem 4.1

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

\[ |B| = |B \cap I| + |B \setminus I| \]
\[ \leq |I \setminus J| + |B \setminus I| \]
\[ < |J \setminus I| + |B \setminus I| \leq |B'| \]

which is a contradiction. The last inequality on the right follows since $J \setminus I \subseteq B'$, and $B \setminus I \subseteq B'$ implies that $J \setminus I \cup B \setminus I \subseteq B'$, but since $J$ and $B$ are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq B'$. 

...
Theorem 4.1

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):
  \[ |B| = |B \cap I| + |B \setminus I| \]
  \[ \leq |I \setminus J| + |B \setminus I| \]
  \[ < |J \setminus I| + |B \setminus I| \leq |B'| \]
  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$. 

...
Dual of a Matroid

Theorem 4.1
Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.
- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):
  \[ |B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| < |J \setminus I| + |B \setminus I| \leq |B'| \]
  which is a contradiction.
- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So $B'$ is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in $M$, and therefore $I \cup \{v\} \in \mathcal{I}^*$. 
Theorem 4.2

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (7)

Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. \textit{i.e.,} $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
Theorem 4.2

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified as follows, for \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)
\]

Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.

Non-negativity integral follows since

\[
|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).
\]

The right inequality follows since \( r_M \) is submodular.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$ (7)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (7)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, $r_{M^*}$ is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (7)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (8)
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (7)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (8)

or

$$r_M(V \setminus X) = r_M(V)$$  \hspace{1cm} (9)
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (7)

**Proof.**

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (8)

or

$$r_M(V \setminus X) = r_M(V)$$  \hspace{1cm} (9)

But a subset $X$ is independent in $M^*$ only if $V \setminus X$ is spanning in $M$ (by the definition of the dual matroid).
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
- It consists of all sets of edges the complement of which contains a spanning tree.
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
- It consists of all sets of edges the complement of which contains a spanning tree.

A graph $G$

Minimally spanning in $M$ (and thus a base)

Minimally spanning in $M^*$ (and thus a base)

Spanning in $M$

Independent in $M^*$
Let $\mathcal{I}$ be a set of subsets of $E$ that is down-closed. Consider a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.

Greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such $y$ exists.

**Theorem 5.1**

Let $\mathcal{I}$ be a non-empty collection of subsets of a set $E$, down-closed (i.e., an independence system). Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$. 

Prof. Jeff Bilmes
EE595A/Spr 2011/Submodular Functions – Lecture 5 - April 13th, 2011
Recall: Matroids by bases

**Theorem 5.2**

**Matroid (by bases)** Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
Matroid and the greedy algorithm

proof of Theorem 5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathbb{R}_+\) is given.
Matroid and the greedy algorithm

proof of Theorem 5.1.

1. Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathbb{R}_+\) is given.
2. Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
proof of Theorem 5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.
proof of Theorem 5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.
- We next show that not only is \(w(A) \geq w(B)\) but that \(w(a_i) \geq w(b_i)\) for all \(i\).
proof of Theorem 5.1.

Assume otherwise, and let \( k \) be the first (smallest) integer such that \( w(a_k) < w(b_k) \).
proof of Theorem 5.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$.
- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$. 
proof of Theorem 5.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$.
- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.
- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$. 

...
proof of Theorem 5.1.

- Assume otherwise, and let \( k \) be the first (smallest) integer such that \( w(a_k) < w(b_k) \).
- Define independent sets \( A_{k-1} = \{a_1, \ldots, a_{k-1}\} \) and \( B_k = \{b_1, \ldots, b_k\} \).
- Since \( |A_{k-1}| < |B_k| \), \( A_{k-1} \cup \{b_i\} \in \mathcal{I} \) for some \( 1 \leq i \leq k \).
- But \( w(b_i) \geq w(b_k) > w(a_k) \), and so the greedy algorithm would have chosen \( b_i \) rather than \( a_k \), contradicting what greedy does.
converse proof of Theorem 5.1.

- Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \((E, I)\) is a matroid.
converse proof of Theorem 5.1.

- Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \((E, I)\) is a matroid.
- Down monotonicity already holds (since we’ve started with an independence system).
converse proof of Theorem 5.1.

- Given an independence system \((E, \mathcal{I})\), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \((E, \mathcal{I})\) is a matroid.

- Down monotonicity already holds (since we’ve started with an independence system).

- Let \(I, J \in \mathcal{I}\) with \(|I| < |J|\). Suppose to the contrary, that \(I \cup \{z\} \notin \mathcal{I}\) for all \(z \in J \setminus I\).
converse proof of Theorem 5.1.

- Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \((E, I)\) is a matroid.

- Down monotonicity already holds (since we’ve started with an independence system).

- Let \(I, J \in I\) with \(|I| < |J|\). Suppose to the contrary, that \(I \cup \{z\} \notin I\) for all \(z \in J \setminus I\).

- Define the following modular weight function \(w\) on \(V\), and define \(k = |I|\).

\[
w(v) = \begin{cases} 
  k + 2 & \text{if } v \in I, \\
  k + 1 & \text{if } v \in J \setminus I, \\
  0 & \text{if } v \in S \setminus (I \cup J)
\end{cases}
\]  

(10)
Matroid and the greedy algorithm

converse proof of Theorem 5.1.

- Now greedy will clearly, after $k$ iterations recover $I$, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2)$. 

On the other hand, $J$ has weight $w(J) \geq |J| (k + 1) > k(k + 2)$ (11) so $J$ has strictly larger weight but is still independent, contradicting greedy's optimality.

Therefore, $(E, I)$ must be a matroid.
converse proof of Theorem 5.1.

- Now greedy will clearly, after $k$ iterations recover $I$, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2)$.

- On the other hand, $J$ has weight
  \[ w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2) \]  
  so $J$ has strictly larger weight but is still independent, contradicting greedy’s optimality.
converse proof of Theorem 5.1.

- Now greedy will clearly, after $k$ iterations recover $I$, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2)$.

- On the other hand, $J$ has weight
  \[ w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2) \] (11)
  so $J$ has strictly larger weight but is still independent, contradicting greedy’s optimality.

- Therefore, $(E, \mathcal{I})$ must be a matroid.
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. 

This is called the restriction of $M$ to $Y$, and is often written $M|_Y$.

If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \}$$

is considered a deletion of $X$ from $M$, and is often written $M \setminus X$. The rank function is of the same form. I.e., $r_Y: 2^Y \rightarrow \mathbb{Z}$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$. 

Prof. Jeff Bilmes
EE595A/Spr 2011/Submodular Functions – Lecture 5 - April 13th, 2011
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then
\[
\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} 
\]
is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the restriction of $M$ to $Y$, and is often written $M|_Y$. 

\[(12)\]
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then
\[ \mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \} \tag{12} \]
is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the \textit{restriction} of $M$ to $Y$, and is often written $M|Y$.

If $Y = V \setminus X$, then we have
\[ \mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \} \tag{13} \]
is considered a \textit{deletion} of $X$ from $M$, and is often written $M \setminus Z$. 
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

(12)

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the restriction of $M$ to $Y$, and is often written $M|_Y$.

If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\}$$

(13)

is considered a deletion of $X$ from $M$, and is often written $M \setminus Z$.

The rank function is of the same form. I.e., $r_Y : 2^Y \rightarrow \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$. 
Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$. 

Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/\ Z$ iff $I \cup X$ is independent in $M$. 

We can see that $M/\ Z = (\ M^* \setminus Z \ )^*$. 

The rank function takes the form $r_{M/\ Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X|Z)$ (14) (see equations 57-60 from lecture 2). 

A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$.

- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$. 

\[ r_{M/Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X | Z) \] (see equations 57-60 from lecture 2).
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** $Z$ is written $M/Z$.

- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.

- We can see that $M/Z = (M^* \setminus Z)^*$.
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$.
- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.
- We can see that $M/Z = (M^* \setminus Z)^*$.
- The rank function takes the form
  \[ r_{M/Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X|Z) \]  
  (see equations 57-60 from lecture 2).
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** \( Z \) is written \( M/Z \).
- Let \( Z \subseteq V \) and let \( X \) be a base of \( Z \). Then a subset \( I \) of \( V \setminus Z \) is independent in \( M/Z \) iff \( I \cup X \) is independent in \( M \).
- We can see that \( M/Z = (M^* \setminus Z)^* \).
- The rank function takes the form

\[
r_{M/Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X | Z) \tag{14}
\]

(see equations 57-60 from lecture 2).
- A **minor** of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid Intersection

Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$. 

**Theorem 6.1**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the maximum size set in $I_1 \cap I_2$ is given by

$$\min_{X \subseteq V} (r_1(X) + r_2(V \setminus X)) \quad (15)$$

In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} (r_1(X) + r_2(Y \setminus X)) \quad (16)$$
Matroid Intersection

- Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$.

- While $(V, I_1 \cap I_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$. 

Theorem 6.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the maximum size set in $I_1 \cap I_2$ is given by:

$$\min_{X \subseteq V} (r_1(X) + r_2(V \setminus X))$$

In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} (r_1(X) + r_2(Y \setminus X))$$
Matroid Intersection

- Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$.

- While $(V, I_1 \cap I_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$.

Theorem 6.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the maximum size set in $I_1 \cap I_2$ is given by

$$\min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right)$$

(15)
Matroid Intersection

- Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$.

- While $(V, I_1 \cap I_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$.

**Theorem 6.1**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. then the maximum size set in $I_1 \cap I_2$ is given by

$$\min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right)$$

(15)

In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} \left( r_1(X) + r_2(Y \setminus X) \right)$$

(16)
Matroid Union

Definition 6.2

Let \( M_1 = (V_1, \mathcal{I}_1) \), \( M_2 = (V_2, \mathcal{I}_2) \), \ldots, \( M_k = (V_k, \mathcal{I}_k) \) be matroids. We define the union of matroids as

\[ M_1 \cup M_2 \cup \cdots \cup M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_k), \]

where

\[
\mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_k = \{ l_1 \cup l_2 \cup \cdots \cup l_k | l_1 \in \mathcal{I}_1, \ldots, l_k \in \mathcal{I}_k \} \quad (17)
\]
**Definition 6.2**

Let $M_1 = (V_1, I_1), M_2 = (V_2, I_2), \ldots, M_k = (V_k, I_k)$ be matroids. We define the union of matroids as $M_1 \vee M_2 \vee \cdots \vee M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, I_1 \vee I_2 \vee \cdots \vee I_k)$, where

$$I_1 \vee I_2 \vee \cdots \vee I_k = \{l_1 \cup l_2 \cup \cdots \cup l_k | l_1 \in I_1, \ldots, l_k \in I_k\} \quad (17)$$

**Theorem 6.3**

Let $M_1 = (V_1, I_1), M_2 = (V_2, I_2), \ldots, M_k = (V_k, I_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left( |Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \right) \quad (18)$$

for any $Y \subseteq V_1 \cup \ldots V_k$. 
Sources for Today’s Lecture