

EE595A – Submodular functions, their optimization and applications – Spring 2011

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Announcements

- Weekly Office Hours: Wednesdays, 12:30-1:30pm, 10 minutes after class on Wednesdays.

Scratch Paper

Scratch Paper

Scratch Paper

Submodular Definitions

Definition (submodular)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (1)$$

An alternate and equivalent definition is:

Definition (diminishing returns)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A \subseteq B \subseteq V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \quad (2)$$

This means that the incremental “value”, “gain”, or “cost” of v decreases (diminishes) as the context in which v is considered grows from A to B .

Ground set: E or V ?

Submodular functions are functions defined on subsets of some finite set, called the **ground set**.

- It is common in the literature to use either E or V as the ground set.

Ground set: E or V ?

$$A, B, C \subseteq E$$

$$x, y \subseteq E \text{ or } V$$

Submodular functions are functions defined on subsets of some finite set, called the **ground set**.

- It is common in the literature to use either E or V as the ground set.
- We will follow this inconsistency in the literature and will inconsistently use either E or V as our ground set (hopefully not in the same equation, if so, please point this out).

Notation \mathbb{R}^E

$$E = [n] = \{1, 2, \dots, n\}$$

$$\mathbb{R}^E = \{x = (x_j \in \mathbb{R} : j \in E)\} \quad (3)$$

$$\mathbb{R}_+^E = \{x = (x_j : j \in E) : x \geq 0\} \quad (4)$$

Any vector $x \in \mathbb{R}^E$ can be treated as a modular function, and vice versa.
That is

$$x(A) = \sum_{a \in A} x_a \quad A \subseteq E \quad (5)$$

$f(A)$

$$\Rightarrow x(\emptyset) = 0$$

Normalized.

Other Notation: indicator vectors of sets

Given an $A \subseteq E$, define the vector $\mathbf{1}_A \in \mathbb{R}_+^E$ to be

$$\mathbf{1}_A(j) = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{if } j \notin A \end{cases} \quad (6)$$

Sometimes this will be written as $\chi_A \equiv \mathbf{1}_A$.

Other Notation: indicator vectors of sets

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Sometimes this will be written as $\chi_A \equiv \mathbf{1}_A$.

Thus, given modular function $x \in \mathbb{R}^E$, we can write $x(A)$ in a variety of ways, i.e.,

$$x(A) = x \cdot \mathbf{1}_A = \sum_{i \in A} x(i) \quad (7)$$

Other Notation: singletons and sets

When A is a set and k is a singleton (a ~~set with~~ single item), the union is properly written as $A \cup \{k\}$, but sometimes I will write just $A + k$.

Summing Submodular Functions

Given E , let $f_1, f_2 : 2^E \rightarrow \mathbb{R}$ be two submodular functions. Then

$$f : 2^E \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A) + f_2(A) \quad (8)$$

is submodular. This follows easily since

$$f(A) + f(B) = f_1(A) + f_2(A) + f_1(B) + f_2(B) \quad (9)$$

$$\geq f_1(A \cup B) + f_2(A \cup B) + f_1(A \cap B) + f_2(A \cap B) \quad (10)$$

$$= f(A \cup B) + f(A \cap B). \quad (11)$$

I.e., it holds for each component of f in each term in the inequality.

any convex combination also works.
convex

$$f(A) = \alpha_1 f_1(A) + \alpha_2 f_2(A)$$

$$\alpha_1, \alpha_2 \geq 0$$

Summing Submodular and Modular Functions

Given E , let $f_1, m : 2^E \rightarrow \mathbb{R}$ be a submodular and a modular function.

Then

$$f = f_1 - m \quad m' = -m$$

$$f : 2^E \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A) - m(A) \quad (12)$$

is submodular (as is $f(A) = f_1(A) + m(A)$). This follows easily since

$$f(A) + f(B) = f_1(A) - m(A) + f_1(B) - m(B) \quad (13)$$

$$\geq f_1(A \cup B) - m(A \cup B) + f_1(A \cap B) - m(A \cap B) \quad (14)$$

$$= f(A \cup B) + f(A \cap B). \quad (15)$$

That is, the modular component with

$m(A) + m(B) = m(A \cup B) + m(A \cap B)$ never destroys the inequality.

Note of course that if m is modular then so is $-m$.

Restricting Submodular Functions

Given E , let $f : 2^E \rightarrow \mathbb{R}$ be a submodular function. And let $S \subseteq E$ be an arbitrary set. then

$$\overset{\text{fixed}}{f'} : 2^E \rightarrow \mathbb{R} \text{ with } f'(A) = f(A \cap S) \quad (16)$$

is submodular.

Proof.

Given $\underline{A} \subseteq \underline{B} \subseteq E \setminus v$, consider

$$f((A + v) \cap S) - f(A \cap S) \geq f((B + v) \cap S) - f(B \cap S) \quad (17)$$

If $v \notin S$, then both differences on each side are zero. If $v \in S$, then we can consider this

$$f(A' + v) - f(A') \geq f(B' + v) - f(B') \quad (18)$$

with $A' = A \cap S$ and $B' = B \cap S$. Since $A' \subseteq B'$, this holds due to submodularity of f . □

Summing Restricted Submodular Functions

Given E , let $f_1, f_2 : 2^E \rightarrow \mathbb{R}$ be two submodular functions and let S_1, S_2 be two arbitrary fixed sets. Then

$$f : 2^E \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A \cap S_1) + f_2(A \cap S_2) \quad (19)$$

is submodular. This follows easily from the preceding two results.

Summing Restricted Submodular Functions

 f_C

Given E , let $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ be a set of subsets of E , and for each $C \in \mathcal{C}$, let $f_C : 2^E \rightarrow \mathbb{R}$ be a submodular function. Then

$$f : 2^E \rightarrow \mathbb{R} \text{ with } f(A) = \sum_{C \in \mathcal{C}} f_C(A \cap C) \quad (19)$$

is submodular. This property is critical for image processing and graphical models.

Ex: $\mathcal{C} = \text{all pairs} = \{\{u, v\} : u, v \in E\}$

$\mathcal{C} = \text{edges of some graphs.}$

$$p(x) \propto \exp(-f(X(x))) \quad \begin{array}{l} X(x) \subseteq E \\ \text{s.t. } i \in X(x) \text{ if } x_i = 1 \end{array}$$

Max

 R_+ ~ Normalization.

Given V , let $c \in \mathbb{R}^V$ be a given fixed vector. Then

$$f : 2^V \rightarrow \mathbb{R} \text{ where } f(A) = \max_{j \in A} c_j \quad (20)$$

is submodular.

$$\max_{j \in \emptyset} c_j = -\infty$$



Proof.

Consider:

$$\max_{j \in A} c_j + \max_{j \in B} c_j \geq \max_{j \in A \cup B} c_j + \max_{j \in A \cap B} c_j \quad (21)$$

which follows since we have that

$$\max(\max_{j \in A} c_j, \max_{j \in B} c_j) = \max_{j \in A \cup B} c_j \quad (22)$$

and

$$\min(\max_{j \in A} c_j, \max_{j \in B} c_j) \geq \max_{j \in A \cap B} c_j \quad (23)$$

Facility Location

Given V, E , let $c \in \mathbb{R}^{V \times E}$ be a given $|V| \times |E|$ ~~non-negative~~ matrix. Then

$$f : 2^E \rightarrow \mathbb{R}, \text{ where } f(A) = \sum_{i \in V} \max_{j \in A} c_{ij} \quad (24)$$

is submodular. This is ~~the~~ ^a facility location function.

Proof.

We can write $f(A)$ as $f(A) = \sum_{i \in V} f_i(A)$ where $f_i(A) = \max_{j \in A} c_{ij}$ is submodular (max of a i^{th} row vector), so f can be written as a sum of submodular functions. □

Facility/Plant Location *(Uncapacitated)*

- Facility Location is a core problem in operations research and a strong motivation for submodular functions. Key goal is to place “facilities” to supply demand sites as efficiently as possible.
- Let E be a set of possible factory/plant locations, and V is a set of sites needing to be serviced (e.g., cities).
- Let c_{ij} be the “benefit” (e.g., $1/c_{ij}$ is the cost) of servicing city i with plant j .
- Let $m : 2^E \rightarrow \mathbb{R}_+^E$ be a plant construction modular function (vector). E.g., $1/m_j$ is the cost to build a plant at location j .
- Each city needs to be serviced by only one plant but no less than one.
- Define $f(A)$ is the ~~maximum~~ “delivery benefit” plus “construction benefit” when the plants in set A are considered to be constructed.
- We can define $f(A) = m(A) + \sum_{i \in V} \max_{j \in A} c_{ij}$.
- Goal is to find a set A that maximizes $f(A)$ (the benefit) placing a bound on the number of plants A (e.g., $|A| \leq k$).

Log Determinant



Let Σ be an $n \times n$ positive ~~semi~~ definite matrix. Let $V = \{1, 2, \dots, n\} \equiv [n]$ be an index set, and for $A \subseteq V$, let Σ_A be the (square) submatrix of Σ obtained by including only entries in the rows/columns given by A .

Then:

$$f(A) = \log \det(\Sigma_A) \text{ is submodular.} \quad (25)$$

(note: still submodular in the semi-def case as well)

Proof.

Suppose $x \in \mathbf{R}^n$ is multivariate Gaussian, that is

$$p(x) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right) \quad (26)$$

...

Log Determinant

$$f(A) = \log \det(\boldsymbol{\Sigma}_A) \text{ is submodular.} \quad (25)$$

...cont.

Then the (differential) entropy of the r.v. X is given by

$$h(X) = \log \sqrt{|2\pi e \boldsymbol{\Sigma}|} = \log \sqrt{(2\pi e)^n |\boldsymbol{\Sigma}|} \quad (26)$$

and in particular, for a variable subset A ,

$$f(A) = h(X_A) = \log \sqrt{(2\pi e)^{|A|} |\boldsymbol{\Sigma}_A|} \quad (27)$$

Entropy is submodular (conditioning reduces entropy), and moreover

$$f(A) = h(X_A) = m(A) + \log |\boldsymbol{\Sigma}_A| \quad (28)$$

where $m(A)$ is a modular function. □

Concave over non-negative modular

Let $m \in \mathbb{R}_+^E$ be a modular function, and g a concave function over \mathbb{R} . Define $f : 2^E \rightarrow \mathbb{R}$ as

Proof.

$$f(A) = g(m(A))$$



then f is submodular.

Proof.

Given $A \subseteq B \subseteq E \setminus v$, we have $0 \leq a = m(A) \leq b = m(B)$, and $0 \leq c = m(v)$. For g concave, we have $g(a + c) - g(a) \geq g(b + c) - g(b)$, and thus

$$g(m(A) + m(v)) - g(m(A)) \geq g(m(B) + m(v)) - g(m(B)) \quad (29)$$



The converse is true as well. **Exercise: prove this.**

Monotone difference of two functions

Let f and G both be submodular functions on subsets of V and let $(f - g)(\cdot)$ be either monotone increasing or monotone decreasing. Then $h : 2^V \rightarrow R$ defined by

$$h(A) = \min(f(A), g(A)) \quad (30)$$

is submodular.

Proof.

If $h(A)$ agrees with either f or g on **both** X and Y , the result follows since

$$\begin{aligned} f(X) + f(Y) &\geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \\ \stackrel{=}{=} g(X) + g(Y) &\geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \\ \stackrel{=}{=} h(X) + h(Y) & \end{aligned} \quad (31)$$

...

Monotone difference of two functions

Let f and G both be submodular functions on subsets of V and let $(f - g)(\cdot)$ be either monotone increasing or monotone decreasing. Then $h : 2^V \rightarrow R$ defined by

$$h(A) = \min(f(A), g(A)) \quad (30)$$

is submodular.

...cont.

Otherwise, w.l.o.g., $h(X) = f(X)$ and $h(Y) = g(Y)$, giving

$$h(X) + h(Y) = f(X) + g(Y) \geq \underbrace{f(X \cup Y) + f(X \cap Y)}_{g(Y) - f(Y)} + \underbrace{g(Y) - f(Y)}_{(31)}$$

$g(Y) - f(Y) \geq g(X \cup Y) - f(X \cup Y)$

By monotonicity, $\underbrace{f(X \cup Y) + g(Y) - f(Y)}_{(31)} \geq g(X \cup Y)$ giving

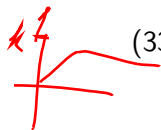
$$h(X) + h(Y) \geq \underbrace{g(X \cup Y) + f(X \cap Y)}_x \geq h(X \cup Y) + h(X \cap Y) \quad (32)$$



Min

Let $f : 2^V \rightarrow \mathbb{R}$ be an increasing or decreasing submodular function and let k be a constant. then the function $h : 2^V \rightarrow \mathbb{R}$ defined by

$$h(A) = \min(k, f(A)) \quad (33)$$



is submodular.

Proof.

For constant k , we have that $(f - k)$ is increasing (or decreasing) so this follows from the previous result.

Note also, $g(a) = \min(k, a)$ for constant k is a concave function, so we can use the earlier result about composing a concave function with a submodular function to get this result as well. □

More on Min

Saturate trick

In general, the minimum of two submodular functions is not submodular. However, when wishing to maximize two monotone non-decreasing submodular functions, we can define function $h : 2^V \rightarrow \mathbb{R}$ as

$$h(A) = \frac{1}{2}(\min(k, f) + \min(k, g)) \quad (34)$$

then h is submodular, and $h(A) \geq k$ if and only if both $f(A) \geq k$ and $g(A) \geq k$.

We plan to revisit this again later in the quarter.

Arbitrary functions as a difference between submodular functions

Given an arbitrary set function f , it can be represented as a difference between two submodular functions $f = g - h$ where both g and h are submodular.

Define

Proof.

Let f be an arbitrary function:

$$\alpha \triangleq \min_{X,Y} \left[\underbrace{f(X) + f(Y)} - \underbrace{f(X \cup Y) - f(X \cap Y)} \right] \quad (35)$$

If $\alpha \geq 0$ then f is submodular, so by assumption $\alpha < 0$. Now let h be an arbitrary submodular function and define *(also nonnegative costs)*

$$\beta \triangleq \min_{X,Y} \left[h(X) + h(Y) - h(X \cup Y) - h(X \cap Y) \right] \quad (36)$$

If $\beta = 0$ then take it to be the smallest positive difference. \Rightarrow

*modular
is not
needed.*

Arbitrary functions as a difference between submodular functions

Given an arbitrary set function f , it can be represented as a difference between two submodular functions $f = g - h$ where both g and h are submodular.

...cont.

Define $f' : 2^V \rightarrow \mathbb{R}$ as

$$f'(A) = f(A) + \frac{|\alpha|}{\beta} h(A) \quad (35)$$

Then f' is submodular, and $f = f'(A) - \frac{|\alpha|}{\beta} h(A)$, a difference between two submodular functions as desired.



Submodular Definitions

Definition (submodular)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (1)$$

An alternate and equivalent definition is:

Definition (diminishing returns)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \quad (2)$$

This means that the incremental “value”, “gain”, or “cost” of v decreases (diminishes) as the context in which v is considered grows from A to B .

Submodular Definitions

An alternate and equivalent definition is:

Definition (group diminishing returns)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $C \subseteq V \setminus B$, we have that:

$$f(A \cup C) - f(A) \geq f(B \cup C) - f(B) \quad (36)$$

This means that the incremental “value” or “gain” of set C decreases as the context in which v is considered grows from A to B (diminishing returns)

Submodular Definitions

Proposition

group diminishing returns implies diminishing returns

Proof.

Obvious, set $C = \{v\}$. □

Submodular Definitions

Proposition

diminishing returns implies group diminishing returns

Proof.

Let $C = \{c_1, c_2, \dots, c_k\}$. Then **diminishing returns** implies yields

$$f(A \cup C) - f(A) \quad (37)$$

$$= f(A \cup C) - \sum_{i=1}^{k-1} \left(f(A \cup \{c_1, \dots, c_i\}) - f(A \cup \{c_1, \dots, c_{i-1}\}) \right) - f(A) \quad (38)$$

$$= \sum_{i=1}^k f(A \cup \{c_1 \dots c_i\}) - f(A \cup \{c_1 \dots c_{i-1}\}) = \sum_i \ell_{c_i}(A \cup \{c_1, \dots, c_{i-1}\}) \quad (39)$$

$$\geq \sum_{i=1}^k f(B \cup \{c_1 \dots c_i\}) - f(B \cup \{c_1 \dots c_{i-1}\}) = \sum \ell_{c_i}(B \cup \{c_1, \dots, c_{i-1}\}) \quad (40)$$

$$= f(B \cup C) - \sum_{i=1}^{k-1} \left(f(B \cup \{c_1, \dots, c_i\}) - f(B \cup \{c_1, \dots, c_{i-1}\}) \right) - f(B) \quad (41)$$

$$= f(B \cup C) - f(B) \quad (42)$$

Submodular Definitions are equivalent

Proposition

The two aforementioned definitions of submodularity submodular and diminishing returns are identical.

Submodular Definitions are equivalent

Proof.

Assume **submodular**. Assume $A \subset B$ as otherwise trivial.

Let $B \setminus A = \{v_1, v_2, \dots, v_k\}$ and define $A^i = A \cup \{v_1 \dots v_i\}$, so $A^0 = A$.

Then by **submodular**,

$$A^k = B$$

$$f(A^i + v) + f(A^i + v_{i+1}) \geq f(A^i + v + v_{i+1}) + f(A^i) \quad (43)$$

or

$$f(A^i + v) - f(A^i) \geq f(A^i + v_{i+1} + v) - f(A^i + v_{i+1}) \quad (44)$$

we apply this inductively, and use

$$f(A^{i+1} + v) - f(A^{i+1}) = f(A^i + v_{i+1} + v) - f(A^i + v_{i+1}) \quad (45)$$

and that $A^{k-1} + v_k = B$.

...

Submodular Definitions are equivalent

...cont.

Assume **group diminishing returns**. Assume $A \neq B$ otherwise trivial.
 Define $A' = A \cap B$, $C = A \setminus B$, and $B' = B$. Then

$$f(A' + C) - f(A') \geq f(B' + C) - f(B') \quad (46)$$

giving

$$f(A' + C) + f(B') \geq f(B' + C) + f(A') \quad (47)$$

or

$$f(A \cap B + A \setminus B) + f(B) \geq f(B + A \setminus B) + f(A \cap B) \quad (48)$$

which is the same as

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (49)$$

Submodular Definitions

Definition (singleton)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A \subset V$, and any $a, b \in V \setminus A$, we have that:

$$f(A \cup \{a\}) + f(A \cup \{b\}) \geq f(A \cup \{a, b\}) + f(A) \quad (50)$$

This follows immediately from **diminishing returns**. To achieve **diminishing returns**, assume $A \subset B$ with $B \setminus A = \{b_1, b_2, \dots, b_k\}$. Then

$$f(A + a) - f(A) \geq f(A + b_1 + a) - f(A + b_1) \quad (51)$$

$$\geq f(A + b_1 + b_2 + a) - f(A + b_1 + b_2) \quad (52)$$

$$\geq \dots \quad (53)$$

$$\geq f(A + b_1 + \dots + b_k + a) - f(A + b_1 + \dots + b_k) \quad (54)$$

$$= f(B + a) - f(B) \quad (55)$$

Gain

It is often the case that we wish to express the gain of an item $j \in V$ in some context, say A , namely $f(A \cup \{j\}) - f(A)$. This is used so often, that there are equally as many ways to notate this. I.e.,

$$f(A \cup \{j\}) - f(A) \stackrel{\Delta}{=} \rho_j(A) \quad (56)$$

$$\stackrel{\Delta}{=} \rho_A(j) \quad (57)$$

$$\stackrel{\Delta}{=} f(\{j\} | A) \quad (58)$$

$$\stackrel{\Delta}{=} f(j | A) \quad (59)$$

We'll use either $\rho_j(A)$ or $f(j|A)$. Note, **diminishing returns** can now be stated as saying that $\rho_j(A)$ is a monotone non-increasing function of A , since $\rho_j(A) \geq \rho_j(B)$ whenever $B \supseteq A$.


Equivalent Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq E \quad (60)$$

$$\rho_j(S) \geq \rho_j(T), \quad \forall S \subseteq T \subseteq E, \text{ with } j \in E \setminus T \quad (61)$$

$$\rho_j(S) \geq \rho_j(S \cup \{k\}), \quad \forall S \subseteq E \text{ with } j \in E \setminus (S \cup \{k\}) \quad (62)$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} \rho_j(S) - \sum_{j \in S \setminus T} \rho_j(S \cup T - \{j\}), \quad \forall S, T \subseteq E \quad (63)$$

S  *T*

- [f(S ∪ T) - f(S ∪ T - j)]

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} \rho_j(S), \quad \forall S \subseteq T \subseteq E \quad (64)$$

$$f(T) \leq f(S) + \sum_{j \in S \setminus T} \rho_j(S \setminus \{j\}), \quad \forall T \subseteq S \subseteq E \quad (65)$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} \rho_j(S \setminus \{j\}) + \sum_{j \in T \setminus S} \rho_j(S \cap T) \quad \forall S, T \subseteq E \quad (66)$$

Equivalent Definitions of Submodularity

We've already seen that Eq. 60 \equiv Eq. 61 \equiv Eq. 62. We next show that Eq. 62 \Rightarrow Eq. 63 \Rightarrow Eq. 64 \Rightarrow Eq. 62.

Eq. 62 \Rightarrow Eq. 63

Let $T \setminus S = \{j_1, \dots, j_r\}$ and $S \setminus T = \{k_1, \dots, k_q\}$.

First *upper bound T in context of S*

$$f(S \cup T) - f(S) = \sum_{t=1}^r [f(S \cup \{j_1, \dots, j_t\}) - f(S \cup \{j_1, \dots, j_{t-1}\})] \quad (67)$$

$$= \sum_{t=1}^r \rho_{j_t}(S \cup \{j_1, \dots, j_{t-1}\}) \leq \sum_{t=1}^r \rho_{j_t}(S) \quad (68)$$

$$= \sum_{j \in T \setminus S} \rho_j(S) \quad (69)$$

Eq. 62 \Rightarrow Eq. 63

Let $T \setminus S = \{j_1, \dots, j_r\}$ and $S \setminus T = \{k_1, \dots, k_q\}$.

Next,

Lower bound S in context of T

$$f(S \cup T) - f(T) = \sum_{t=1}^q [f(T \cup \{k_1, \dots, k_t\}) - f(T \cup \{k_1, \dots, k_{t-1}\})] \quad (67)$$

$$= \sum_{t=1}^q \rho_{k_t}(T \cup \{k_1, \dots, k_t\} \setminus \{k_t\}) \geq \sum_{t=1}^q \rho_{k_t}(T \cup S \setminus \{k_t\}) \quad (68)$$

$$= \sum_{j \in S \setminus T} \rho_j(S \cup T \setminus \{j\}) \quad (69)$$

Eq. 62 \Rightarrow Eq. 63

Let $T \setminus S = \{j_1, \dots, j_r\}$ and $S \setminus T = \{k_1, \dots, k_q\}$.

So we have

Upper bound

$$f(S \cup T) - f(S) \leq \sum_{j \in T \setminus S} \rho_j(S) \quad (67)$$

and

Lower bound

$$f(S \cup T) - f(T) \geq \sum_{j \in S \setminus T} \rho_j(S \cup T \setminus \{j\}) \quad (68)$$

and subtracting the 2nd from the first gives the result.

$$(68) - (67)$$

$$f(T) - f(S) \leq \dots$$

Eq. 63 \Rightarrow Eq. 64

This follows immediately since if $S \subseteq T$, then $S \setminus T = \emptyset$, and the last term of Eq. 63 vanishes.

Eq. 64 \Rightarrow Eq. 62

Here, we set $T = S \cup \{j, k\}$, $j \notin S \cup \{k\}$ into Eq. 64 to obtain

$$f(S \cup \{j, k\}) \leq f(S) + \rho_j(S) + \rho_k(S) \quad (69)$$

$$= f(S) + f(S + \{j\}) - f(S) + f(S + \{k\}) - f(S) \quad (70)$$

$$= f(S + \{j\}) + f(S + \{k\}) - f(S) \quad (71)$$

$$= \rho_j(S) + f(S + \{k\}) \quad (72)$$

giving

$$\rho_j(S \cup \{k\}) = f(S \cup \{j, k\}) - f(S \cup \{k\}) \quad \checkmark \quad (73)$$

$$= f(S \cup \{j, k\}) - \rho_k(S) - f(S) \quad \checkmark \quad (74)$$

$$\leq \rho_j(S) \quad (75)$$

Sources for Today's Lecture

Lovasz-1983, Nemhauser, Wilsey, Fisher-1978, Narayanan-1997, and Narasimhan, Bilmes-2005.