EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 20 - June 9th, 2011
Announcements

- Final lecture.
We need to find one makeup lecture this term.

L10 (5/4):   L20: (6/9): Sym SFM, sub. max,
Submodular Function Minimization

- Wolfe (1976)/von Hohenbalken (1975)
- Edmonds (1965)
- Bixby, Cunningham, Topkis (1984)
- Cunningham (1985)
- Schrijver (2000)
- Fleischer, Iwata (2000)
- Iwata (2002)
- Iwata (2003)
- Orlin (2007)
- Iwata, Fleischer, Fujishige (2000)
- Iwata, Orlin (2009)

- Ellipsoid Method: $O(n^5 \gamma \log M)$
- $O(n^7 \gamma \log n)$
- $O(n^7 \gamma + n^8)$
- $O((n^4 \gamma + n^5) \log M)$
- $O(n^5 \gamma + n^6)$
Recall extended polymatroid which applies to any normalized submodular function

$$EP_f = \{ x : x(A) \leq f(A), \forall A \subseteq E \}$$  \hspace{1cm} (1)
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\[ EP_f = \{ x : x(A) \leq f(A), \forall A \subseteq E \} \]  

(1)

Let \( f \) be submodular and let \( a \in \mathbb{R}^E \). Define function \( f_a \) on \( E \) as

\[ f_a(B) = \min_{A \subseteq B} (f(A) + a(B \setminus A)) \]  

(2)

for any \( B \subseteq E \).
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for any \( B \subseteq E \).

- Then \( f_a \) is submodular. Moreover,

\[ EP_{f_a} = \{ x \in EP_f : x \leq a \} \quad (3) \]

and

\[ P_{f_a} = \{ x \in P_f : x \leq a \} \quad (4) \]
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Therefore, if \( P \) is an extended polymatroid, then so is \( P \cap \{ x : x \leq a \} \).
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Therefore, if \( P \) is an extended polymatroid, then so is \( P \cap \{ x : x \leq a \} \).

We also saw earlier that for any \( EP_f \),

\[ f(A) = \max \{ x(A) : x \in EP_f \} \] (5)

for any \( A \subseteq E \).
Now, take $a = 0$, and we get

\[ f_0(E) = \min_{A \subseteq E} f(A) \]  \hspace{1cm} (6)

\[ EP_{f_0} = EP_f \cap \{x : x \leq 0\} \]  \hspace{1cm} (7)
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We see
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\min \left\{ \sum_{e \in E} x(e)^2 : x \in B_f \right\}
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Given solution \(\hat{x}\) to the above, minimal/maximal solution becomes

\(A_- = \{ e : \hat{x}(e) < 0 \}\) and \(A_0 = \{ e : \hat{x}(e) \leq 0 \}\).
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although complexity is currently unknown.
We have that $f(x)$ is convex, $f_b(x)$ is linear, and is a tight lower bound on $f(x)$.
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Can there be both a tight linear upper bound and tight linear lower bound on a convex function, where each bound is tight at the same point?
Submodular Functions and Tight Linear Lower Bounds

- We saw that given any submodular function $f$, and a base $x \in B_f$ generated by the greedy algorithm on the ordered set $E_n = (e_1, e_2, \ldots, e_n)$, we have

- $x$ is tight at each of $E_i$. That is $x(E_i) = f(E_i)$

- Also since $x \in P_f$, then $x(A) \leq f(A)$ $\forall A$. 
Recall from lecture 2 (equation 63)

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} \rho_j(S) - \sum_{j \in S \setminus T} \rho_j(S \cup T - \{j\}), \quad \forall S, T \subseteq E \]  

Thus, we can define a tight modular upper bound. Given \( S \subseteq E \),

\[ h_S(T) = f(S) + \sum_{j \in T \setminus S} \rho_j(S) - \sum_{j \in S \setminus T} \rho_j(S \cup T - \{j\}) = f(S) + \sum_{j \in S \cap T} \rho_j(S \cup T - \{j\}) \]  

Then \( h_S \) is modular, \( h_S(S) = f(S) \), and \( f(T) \leq h_S(T) \), \( \forall T \).
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Using submodular diminishing returns, we can weaken this to

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} \rho_j(S) - \sum_{j \in S \setminus T} \rho_j(E \setminus \{j\}), \forall S, T \subseteq E \quad (11) \]
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    & = f(S) + \sum_{j \in T \setminus S} \rho_j(S) - \sum_{j \in S} \rho_j(E - \{j\}) + \sum_{j \in S \cap T} \rho_j(E - \{j\}) \\
    & = \text{const}_S + \text{modular}(T) 
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Then \( h_S \) is modular, \( h_S(S) = f(S) \), and \( f(T) \leq h_S(T) \ \forall T \).
If $f$ is both submodular, and also we have that $f(A) = f(E \setminus A)$ for all $A$, then $f$ is said to be symmetric submodular.
Minimization of a Symmetric Submodular Functions

- If \( f \) is both submodular, and also we have that \( f(A) = f(E \setminus A) \) for all \( A \), then \( f \) is said to be **symmetric submodular**

- Given any non-symmetric submodular function \( g \), we can always symmetrize it, \( f(A) = g(A) + g(E \setminus A) - g(E) \). Then \( f \) is symmetric and normalized \( f(\emptyset) = 0 \).
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- Such an $f$ is also non-negative since

$$2f(A) = f(A) + f(E \setminus A) \geq f(\emptyset) + f(E) = 2f(\emptyset) \geq 0 \quad (14)$$
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Such a symmetrized submodular function measures the “dependence” between $A$ and $E \setminus A$. If $f(A) = 0$ then $g$ is decomposable as $g(B) = g(B \cap A) + g(B \cap (E \setminus A))$. 
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- Example: $g =$ entropy, then $f =$ mutual information.

- In the following slides, $f$ will be presumed to be the symmetrized version of $g$. 
Minimizing symmetric submodular functions can be done in strongly polynomial time $O(n^3)$. The algorithm is by Nagamochi & Ibaracki 1992 for graph cuts and was shown by Queyranne in 1995 to work for sym. SFM.
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The algorithm first finds a MA (maximum adjacency) or a maximum back order.

1. Choose \( v_1 \) arbitrarily;
2. \( W_1 \leftarrow (v_1) \);
3. for \( i \leftarrow 1 \ldots |V| - 1 \) do
   4. Choose \( v_{i+1} \in \text{argmin}_{u \in V \setminus W_i} \left( g(W_i \cup \{u\}) - g(u) \right) \);
   5. \( W_{i+1} \leftarrow (W_i, v_{i+1}) \); /* Append \( v_{i+1} \) to end of \( W_i \)*/
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Note algorithm operates on non-symmetric function $g$. If $g$ is already symmetric, then $f = g$. 
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- Note algorithm operates on non-symmetric function $g$. If $g$ is already symmetric, then $f = g$.
- The final ordered set $W_n = (v_1, v_2, \ldots, v_n)$ is special in that the last two nodes $(v_{n-1}, v_n)$ serve as a surrogate minimizer for a special case.
Pendent pair

- A ordered pair of elements \((t, u)\) is called a pendent pair if \(u\) is a minimizer amongst all sets that separate \(u\) and \(t\).
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*In the ordered set \(W = (v_1, \ldots, v_n)\) generated by the MA algorithm, then \((v_{n-1}, v_n)\) is a pendent pair.*

- Interestingly, this algorithm is the same as maximum cardinality search (MCS), when \(f\) represents a graph cut function (recall, MCS is used to efficiently test graph chordality).
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We store the score of the first case, and consider a new element \(tu\) and clustered ground set \(V' = V \setminus \{t, u\} \cup \{tu\}\), and new symmetric submodular function \(f' : 2^{V'} \rightarrow \mathbb{R}\) with

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f'(X) = \begin{cases} 
  f(X) & \text{if } tu \notin X \\
  f(X \cup \{t, u\} \setminus \{tu\}) & \text{if } tu \in X 
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We then find a new pendent pair on \(f'\) using the above algorithm, store the min value, and merge.

We do this \(n\) times. We take the min over all of the stored values.

The pendent pair corresponding to the min element, say \((t', u')\) might actually correspond to clusters, so we use the original ground elements corresponding to \(u'\).
Theorem 5.2

The final resultant $u'$ when expanded to original ground elements minimizes the symmetric submodular function $f$ in $O(n^3)$ time.

- This has become known as Queyranne’s algorithm for symmetric submodular function minimization.
- This was done in 1995 and it is said that this result, at that time, rekindled the efforts to find general combinatorial SFM.
Maximization of Submodular Functions

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- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).
The Set Cover Problem

- Let $E$ be a round set and let $E_1, E_2, \ldots, E_m$ be a set of subsets.
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- The set cover problem asks for the smallest subset \( X \) of \( V \) such that \( f(X) = |E| \) (smallest subset of the subsets of \( E \)) where \( E \) is still covered. I.e.,

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\min |X| \text{ subject to } f(X) \geq |E| \quad (18)
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- We might wish to use a more general modular function $m(X)$ rather than cardinality $|X|$.
- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - \epsilon) \log n$ unless NP is slightly superpolynomial \((n^{O(\log \log n)})\).
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- The max $k$ cover problem asks, given a $k$, what sized $k$ set of sets $X$ can we choose that covers the most? I.e., that maximizes $f(X)$ as in:

$$\max f(X) \text{ subject to } |X| \leq k$$  \hspace{1cm} (19)
The Max k-Cover Problem

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An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
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Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \text{argmax} \ f(S_i \cup \{v\}) \right\}$$  \hspace{1cm} (20)
Cardinality Constrained Max. of Polymatroid Functions

- This algorithm has a guarantee

**Theorem 6.1**

*Given a polymatroid function $f$, the above greedy algorithm returns sets $S_i$ such that for each $i$ we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.*

- To find $A^* \in \text{argmax} \{ f(A) : |A| \leq k \}$, we repeat the greedy step until $k = i + 1$:
- Again, since this generalizes max $k$-cover, Feige (1998) showed that this can’t be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$. 
Minimum Submodular Cover

- Given polymatroid $f$, goal is to find a covering set of minimum cost. That is:

$$S^* \in \arg\min_S |S| \text{ such that } f(S) \geq \alpha$$

where $\alpha$ is a “cover” requirement.
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- For integer valued $f$, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where $\alpha$ is the desired cover constraint.
Generalizations

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- We may wish to do both.
- In either case, the only hope is approximation algorithms. Question is, what is the tradeoff between running time and approximation quality, and is it possible to get tight bounds (i.e., an algorithm that achieves an approximation ratio, and a proof that one can’t do better than that unless some extremely unlike event were to be true, such as P=NP).
Submodular max with constraints

- Consider a set of sets $\mathcal{I}$ and one wishes to find $\max \{ f(A) : A \in \mathcal{I} \}$
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- In fact, if $M = (E, \mathcal{I})$ is a $k$-uniform matroid we saw in Lecture 3 ($\mathcal{I} = \{ A \subseteq E : |A| \leq k \}$), we have the cardinality constrained submodular max we just encountered.
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- In fact, one could consider multiple matroids $M_1, M_2, \ldots, M_p$ and one may want a solution that is independent in all matroids. I.e., the constraint set is that $A \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_p$
Greedy over multiple matroids

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- I.e., Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \arg\max_{v \in V \setminus S_i : S_i + v \in \bigcap_{i=1}^{P} I_i} f(S_i \cup \{v\}) \right\}$$

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**Theorem 6.2**

*Given a polymatroid function $f$, and set of matroids $\{M_j = (E, I_j)\}_{j=1}^p$, the above greedy algorithm returns sets $S_i$ such that for each $i$ we have $f(S_i) \geq \frac{1}{p+1} \max_{|S| \leq i, S \in \bigcap_{i=1}^p I_i} f(S)$, assuming such sets exists.*
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assuming such sets exists.

- So this is a very easy algorithm to solve multiple matroid constraints, but the bound is not that good when there are many matroids.
The constraint $|A| \leq k$ is a simple cardinality constraint.
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Monotone Submodular over Knapsack Constraint
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- A knapsack constraint would be of the form $c(A) \leq b$ where $B$ is some integer budget that must not be exceeded. That is $\max \{ f(A) : A \subseteq V, c(A) \leq b \}$.
- $c(e)$ may be seen as the cost of item $e$ and if $c(e) = 1$ for all $e$, then we recover the cardinality constraint we saw earlier.
Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with \( S_0 = \emptyset \), we repeat the following greedy step

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S_{i+1} = S_i \cup \left\{ \arg\max_{v \in V \setminus S_i} \left( f(S_i \cup \{v\}) - f(S_i) \right) \right\}
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the gain is \( f(\{v\}|S_i) = f(S_i + v) - f(S_i) \), so greedy just chooses next the currently unselected element with greatest gain.
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Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set \( S_0 \), we repeat the following cost-normalized greedy step

\[
S_{i+1} = S_i \cup \left\{ \arg\max_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\}
\]  

(24)

which we repeat until \( c(S_{i+1}) > b \) and then take \( S_i \) as the solution.
A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with \( S_0 = \emptyset \), and compare the solution found with the max of the singletons \( \max_{v \in V} f(\{v\}) \), choosing the max, then we get a \((1 - e^{-1/2}) \approx 0.39\) approximation, in \( O(n^2) \) time.
- On the other hand, we can get a \((1 - e^{-1}) \approx 0.63\) approximation in \( O(n^5) \) time if we run the above procedure starting from all sets of cardinality three (so restart for all \( S_0 \) such that \(|S_0| = 3\)), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to \( d \) simultaneous knapsack constraints is possible as well.
Minoux’s Accelerated Greedy for Submodular Functions

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- At stage $i$ in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$.
Greedy can be made much faster by a simple strategy made possible, once again, via the use of submodularity.

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Once we choose a max $v$, then set $S_{i+1} \leftarrow S_i + v$. 

For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity. Therefore, if we find a $v'$ such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since $f(v|S_{i+1}) \leq \alpha_v = f(v|S_i)$, we need not re-evaluate the gain.

Strategy is: find the max $\alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other $\alpha_{v'}$'s then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with real value, resort, and repeat.

In practice, this results in enormous speedups over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue), and can be used for very large data sets (e.g., social networks (selecting blogs of greatest influence), and also document summarization).
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- At stage $i$ in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$.
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- Strategy is: find the max $\alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other $\alpha_v$’s then that’s the next greedy step. Otherwise, replace $\alpha_{v'}$ with real value, resort, and repeat.
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In practice, this results in enormous speedups over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue), and can be used for very large data sets (e.g., social networks (selecting blogs of greatest influence), and also document summarization).
What About Non-monotone

If $f$ is an arbitrary submodular function (so neither polymatroidal, nor necessarily positive or negative), then verifying if the maximum of $f$ is positive or negative is already NP-hard.
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- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.

- It is possible to get a \( \left( \frac{1}{3} - \frac{\epsilon}{n} \right) \) approximation for maximizing non-monotone non-negative submodular functions, using an algorithm that uses at most \( O\left( \frac{1}{\epsilon} n^3 \log n \right) \) function calls.
Submodularity and local optima

Given any submodular function $f$, a set $S \subseteq V$ is a local maximum of $f$ if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$. 

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Prof. Jeff Bilmes
EE595A/Spr 2011/Submodular Functions – Lecture 20 - June 9th, 2011
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- The following interesting result is true for any submodular function:
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Given a submodular function $f$, if $S$ is a local optimum of $f$, and $I \subseteq S$ or $I \supseteq S$, then $f(I) \leq f(S)$. 
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- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.
More general still: multiple constraints different types

- In the past several years, there has been a plethora of papers on maximizing both monotone and non-monotone submodular functions under various combinations of one or more knapsack and/or matroid constraints.
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- The approximation quality is usually some function of the number of matroids, and is often not a function of the number of knapsacks.
- Often the computational costs of the algorithms are prohibitive (e.g., exponential in $k$) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.
Submodular Max and polyhedral approaches

- We’ve spent much time discussing SFM and the polymatroidal polytope, and in general polyhedral approaches for SFM.
- Most of the approaches for submodular max have not used such an approach.
- Very recently, a paper by Chekuri, Vondrak, and Zenklusen (2011, appeared yesterday in fact) make some progress on this front using multilinear extensions.
Definition 6.4

For a set function $f : 2^V \to \mathbb{R}$, define its multilinear extension $F : [0, 1]^V \to \mathbb{R}$ by

$$F(x) = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j)$$

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- If \( f \) is submodular, then \( \frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0 \) for all \( i, j \in V, x \in [0, 1]^V \).
Multilinear extension

Moreover, we have

\[ F(x) = \sum_{i=1}^{n} f_i(x_i) \]

If \( f \) is monotone non-decreasing, then \( F \) is non-decreasing along any line of direction \( d \in \mathbb{R}^E \) with \( d \geq 0 \).

If \( f \) is submodular, then \( F \) is concave along any line of direction \( d \geq 0 \), and is convex along any line of direction \( 1_v - 1_w \) for any \( v, w \in V \).
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Submodular Max and polyhedral approaches

- Basic idea: Given a set of constraints $\mathcal{I}$, we form a polytope $P_\mathcal{I}$ such that $\{1_I : I \in \mathcal{I}\} \subseteq P_\mathcal{I}$

- We find $\max_{x \in P_\mathcal{I}} F(x)$ where $F(x)$ is the multi-linear extension of $f$, to find a fractional solution $x^*$

- We then round $x^*$ to a point on the hypercube, thus giving us a solution to the discrete problem.
In the recent paper by Chekuri, Vondrak, and Zenklusen, they show:

1) Constant factor approximation algorithm for max \{F(x) : x \in P\} for any down-monotone solvable polytope P and F multilinear extension of any non-negative submodular function.

2) A randomized rounding scheme to obtain an integer solution.

3) An optimal \((1 - 1/e)\) instance of their rounding scheme that can be used for a variety of interesting independence systems, including \(O(k)\) knapsacks, \(k\)-matroids and \(O(1)\) knapsacks, a \(k\)-matchoid and \(\ell\)-sparse packing integer programs, and unsplittable flow in paths and trees.
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**Where next?**

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This course has served as a thorough introduction to many important aspects of submodular functions.
Sources for Today’s Lecture

- Chekuri, Vondrak, Zenklusen, “Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes”, 2011 (a recent paper (appeared yesterday) that, among other things, has a nice up-to-date summary on all the results on submodular max).
- Jegelka & Bilmes, ”Approximate Probabilistic Inference via Generalized Graph Cuts”, 2011.