

EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 17 - May 27th, 2011

Announcements

- Last lecture, and final presentations, will take place Thursday, June 9th, from 3-7:30pm. The lecture will be from 3:00-5:00pm, and the final presentations will be from 5:00-7:30pm. Please bring dinner.

(5/27/2011: Note: I didn't have my tablet for today's lecture, so I made annotations using the popup tool in acrobat. A few of the figures I drew on the board were photographed and included in the "scratch" pages at the end of this lecture).

Class Road Map

We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): lattices, → SFM.
- L16 (5/25): → SFM
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?

dep and partial order

- Consider $x \in P_f$, and consider the following set

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- Moreover, define a partial order on $\text{DEP}(x)$ as follows: if $A, B \in \text{DEP}(x)$, then $A \preceq B$ iff $A \subseteq B$.

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- We're going to use this partial order to define a partial order on all elements of $\text{sat}(x)$.
- Now recall $\mathcal{D}(x) = \{A : x(A) = f(A)\}$ forms a distributive lattice. What is the natural partial order?

dep and partial order

- Now in any distributive lattice L , consider its join-irreducibles \mathcal{J} (i.e., any element $A \in \mathcal{J}$ can't be represented as a join of any other two elements in L).
- We saw that if the lattice has length n , then \mathcal{J} will have exactly n elements (in the Boolean case, these are atoms/ground elements), and each element in \mathcal{J} is partially ordered by the lattice partial order.
- Moreover, we saw any element can be “generated” by joining the join-irreducible elements.

dep and partial order

- Now any element in $\text{DEP}(x)$ (for x extreme) can't be represented by the join of two other elements in $\text{DEP}(x)$, since the minimal tight sets containing e would not be generated by merging two minimal tight sets containing, say, a , and b , where all of a, b, c are unequal.
- Thus, considering $\mathcal{D}(x)$ as a distributed lattice, then $\text{DEP}(x)$ are the join-irreducibles.
- And the order \preceq defined earlier is the natural order w.r.t. this lattice and its join-irreducibles.

dep and partial order

- Let $x \in P_f$ again be an extreme point, and let it be generated by an ordering of $B = (e_1, e_2, \dots, e_k) \subseteq E$ with $B_i = (b_i, b_2, \dots, b_i)$ a partial order w.r.t. ordered items B (B and $B_i, \forall i$ are ordered sets).

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- Recall, the equation for x is of the form $x(e) = 0$ for some e and $x(A) = f(A)$ for some A (see earlier). Specifically, we have that $x(E \setminus B) = 0$ and, for $i = 1 \dots k$, $x(B_i) = f(B_i)$.

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- Thus, each of B_i is a tight set.
- We also have that $\text{supp}(x) \subseteq B$.



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- Thus, each of B_i is a tight set.
- We also have that $\text{supp}(x) \subseteq B$.
- Thus, for any $d, e \in \text{supp}(x) \subseteq B$, there is a tight set containing one but not the other. Specifically, let $d = e_i$ and $e = e_j$ with $j > i$. Then non-zero B_i (i.e., $B_i \cap \text{supp}(x)$) contains d but not e (note, vice versa is not true).

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- Thus, for any $d, e \in \text{supp}(x) \subseteq B$, we have $\text{dep}(x, d) \neq \text{dep}(x, e)$.
- Moreover, for any $e \in B$, we **can** have that $\text{dep}(x, e) = B_i$ where $e = e_i$. *This point is further clarified in the next slide.*

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- For any $j < i$, B_j does not contain e_i .
- Thus, $\text{dep}(x, e_i)$ (minimal tight e_i -containing set) **might** equal B_i .
- On the other hand, consider the extreme vector $x^{(i)} \in \mathbb{R}^E$ with

$$x^{(i)}(e) = \begin{cases} x(e) & \text{if } e \in B_i \\ 0 & \text{else} \end{cases} \quad (2)$$

so $x^{(i)}$ is just the extreme vector generated by the ordered set B_i .

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- On the other hand, $B_i \not\subset \text{dep}(x, e_i)$ due to $\text{dep}(x, e_i)$'s minimality.
- Therefore, we see that in general, $\text{dep}(x, e_i) \subseteq B_i$.

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- Then $\text{dep}(x, e_i) = B_j^{e_i}$ where

$$B^{e_i} \in \underset{B' \in \mathcal{B}(x)}{\text{argmin}} e_i(B') \quad \text{🗨️} \quad (3)$$

is ordered, and j is the position of e_i in B^{e_i} . All follows from iff relationship between extremal points and greedy algorithm.

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- Also, for $d, e \in \text{sat}(x)$, $\text{dep}(x, d) = B_i^d \subset \text{dep}(x, e) = B_j^e$ iff $d \in \text{dep}(x, e)$.

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 - Clearly, $\text{dep}(x, d) \subset \text{dep}(x, e) \Rightarrow d \in \text{dep}(x, e)$.

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- Also, for $d, e \in \text{sat}(x)$, $\text{dep}(x, d) = B_i^d \subset \text{dep}(x, e) = B_j^e$ iff $d \in \text{dep}(x, e)$.
 - Clearly, $\text{dep}(x, d) \subset \text{dep}(x, e) \Rightarrow d \in \text{dep}(x, e)$.
 - Also $d \in \text{dep}(x, e)$ means $\text{dep}(x, d) \subseteq B_k^e$ where $k = d(B^e)$ is the position of d in B^e (since B_k^e is a tight set containing d), but it must be that $k < j$ (since B_j^e is the smallest tight set containing e and the j 'th position of B_j^e is e).

dep and partial order

- Also, for polymatroidal f , we saw earlier that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight. This follows since $x(\text{supp}(x) + e) = x(\text{supp}(x))$ but e is dependent on $\text{supp}(x)$ so that $f(\text{supp}(x) + e) = f(\text{supp}(x))$.

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- Now for any point $a, b \in \text{sat}(x) \setminus \text{supp}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, b)$

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- Now for any point $a, b \in \text{sat}(x) \setminus \text{supp}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, b)$
 - This follows, since the minimal tight set containing a would never contain b (and in this case, vice versa).
 - I.e., in such case, we **can** have for $e \in \text{sat}(x) \setminus \text{supp}(x)$, $\text{dep}(x, e) = B_j + e$ for some j , the smallest j such that $f(B_j + e) = f(B_j)$, and note that $a \notin B_j$.

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- Now for any point $a, b \in \text{sat}(x) \setminus \text{supp}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, b)$
 - This follows, since the minimal tight set containing a would never contain b (and in this case, vice versa).
 - I.e., in such case, we **can** have for $e \in \text{sat}(x) \setminus \text{supp}(x)$, $\text{dep}(x, e) = B_j + e$ for some j , the smallest j such that $f(B_j + e) = f(B_j)$, and note that $a \notin B_j$.
- This gives further support to the phrase “dependence function”, namely $\text{dep}(x, e) \setminus \{e\} = B_j$ is the smallest set that renders e dependent (again, like the fundamental circuit of a matroid).

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- This gives further support to the phrase “dependence function”, namely $\text{dep}(x, e) \setminus \{e\} = B_j$ is the smallest set that renders e dependent (again, like the fundamental circuit of a matroid).
- Thus, we have 1-1 mapping between **all** elements of $\text{sat}(x)$ and $\text{DEP}(x) = \{\text{dep}(x, e) : e \in \text{sat}(x)\}$.

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- Therefore, the partial order on $\text{DEP}(x)$ can be used to define a partial order on $\text{sat}(x)$.

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- Thus, we can define a partial order on the elements of $\text{sat}(x)$ as follows:

Definition 2.1 (partial order on elements of $\text{sat}(x)$)

For $d, e \in \text{sat}(x)$, we have

$$d \preceq e \Leftrightarrow d \in \text{dep}(x, e) \quad (4)$$

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Theorem 3.1

If $x \in P_f$ is an extreme point, then \preceq is a partial order on $\text{sat}(x)$ where for $a, e \in \text{sat}(x)$, the order \preceq is defined by: $a \preceq e$ iff $a \in \text{dep}(x, e)$.

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Theorem 3.2

$x \in P_f$ is an extreme point, iff $\text{supp}(x) \subseteq \text{sat}(x)$ and $\text{dep}(x, a) \neq \text{dep}(x, b)$ for every pair of distinct points $a, b \in \text{sat}(x)$.

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Definition 3.3

Given a partial order \preceq and an ordered set $B = (e_1, e_2, \dots, e_k)$, then B is **compatible** with \preceq if for all $i < j$ we have that $e_i \preceq e_j$.

the partial order of extreme points

Theorem 3.4

Let x be an extreme point of P_f and \preceq be its partial order. Let $B \subseteq E$ be an ordered set. Then B generates x using the greedy algorithm iff we have $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$ and B is compatible with \preceq .

Proof.

- Generate \Rightarrow Compatible: Let B generate x

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- Generate \Rightarrow Compatible: Let B generate x
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- Also, since B is tight, $B \in \mathcal{D}(x)$, so $B \subseteq \text{sat}(x)$.

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- Then $\text{supp}(x) \subseteq B$.
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- Moreover, $B_j \in \mathcal{D}(x)$ (for $1 \leq j \leq |B|$), so that $\text{dep}(x, e_j) \subseteq B_j$ for e_j the j 'th element of B (note $\text{dep}(x, e_j) \subseteq B_j$ if $(\text{sat}(x) \setminus \text{supp}(x)) \cap B_j = \emptyset$).

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- But $g \notin B_j$ means $g \notin \text{dep}(x, e_j)$, which means $g \not\preceq e_j$, meaning B is compatible with \preceq .

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- Thus, B_j is the union of tight sets (since each of $\text{dep}(x, e)$ is tight), so that B_j is also tight (unions of tight sets are tight).


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- Thus, B_j is the union of tight sets (since each of $\text{dep}(x, e)$ is tight), so that B_j is also tight (unions of tight sets are tight).
- Thus B is tight and thus x is generated by the ordering given in B (by the greedy algorithm).



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Corollary 3.5

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$, then x is generated using greedy by some ordering of B .

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- this is a more satisfying way to given an extreme point show that the greedy algorithm can generate it than to resort to the polyhedral $cv = \max\{cx : x \in P_f\}$ for an appropriate direction c .
- Moreover, we can produce an efficient $O(|E|^2)$ algorithm that can produce \preceq for any extreme point x of P_f .
- The algorithm does so by, for each $e \in \text{sat}(x)$, producing the sets $\text{dep}(x, e)$ that define the order (or otherwise terminating by saying that x is not an extreme point).

Generating the partial order of an extreme point

input : Vector $x \in \mathbb{R}^E$, polymatroid function f on E .

output: That x is not extreme point, or if it is, minimal tight sets $\text{dep}(x, e)$ for $e \in \text{sat}(x)$ thus defining \preceq . Moreover, $\text{dep}(x, e_j) = A_j$ for $1 \leq j \leq n$ where $n = |\text{sat}(x)|$.

```

1  $j \leftarrow 0$  ;  $B \leftarrow \emptyset$  ;
2 while true do
3    $j \leftarrow j + 1$  ;
4   if  $\exists e \in E \setminus B$  with  $x(B + e) = f(B + e)$  then
5      $B \leftarrow B + e, e_j \leftarrow e.$ 
6   else
7      $\text{STOP}$ , if  $\text{supp}(x) \subseteq B$  then  $x$  is extreme, otherwise not.
8    $A_j \leftarrow B; k \leftarrow j - 1$  ;
9   while  $x(A_j - e_k) = f(A_j - e_k)$  and  $k > 0$  do
10     $A_j = A_j - e_k; k \leftarrow k - 1$ 

```

On partial order algorithm

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- Thus, extreme point testing is fundamentally computationally simpler than arbitrary membership testing (recall, to test if $x \in P_f$ in general, we needed submodular function minimization).
- To determine, only, if a given x is extreme, we can delete lines 8-10 (having same run time).

Maximal in a tight set

Theorem 3.6

Given an extreme point $x \in P_f$, with A tight for x , and if given order \preceq element $e \in R$ is maximal, then $A - e$ is also tight.

Proof.

- If e is maximal in A w.r.t. \preceq , then there exists **no** $a \in A \setminus \{e\}$, such that $e \in \text{dep}(x, a)$.

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- Now, since $\text{dep}(x, a)$ is the smallest x -tight set **containing** a and $\text{dep}(x, a) \subseteq A \setminus \{e\}$, we have

$$\bigcup_{a \in A \setminus \{e\}} \text{dep}(x, a) = A \setminus \{e\} \quad (5)$$

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$$\bigcup_{a \in A \setminus \{e\}} \text{dep}(x, a) = A \setminus \{e\} \quad (5)$$

- Since the union (and intersection) of tight sets is tight, we have that $A \setminus \{e\}$ is therefore also tight.



Maximal in a tight set

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Given an extreme point $x \in P_f$, with A tight for x , and if given order \preceq element $e \in R$ is maximal, then $A - e$ is also tight.

We also have

Corollary 3.7

For all $e \in \text{sat}(x)$, we have that $\text{dep}(x, e) \setminus e$ is also tight.

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- $\text{dep}(x, e)$ is tight, and recall that there is some ordered set B_j^e with $\text{dep}(x, e) = B_j^e$ whose's last (j 'th) item is e .



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Maximal in a tight set

- Also, for any $x \in P_f$, and $\forall e \in \text{sat}(x)$, we have that

$$\text{dep}(x, e) \setminus \{e\} \subseteq \text{supp}(x) \quad (6)$$

- This follows since suppose $\exists e' \in \text{dep}(x, e) \setminus \{e\}$ such that $x(e') = 0$.
- Then, since $f(e') > 0$, in such case $\text{dep}(x, e)$ wouldn't be minimally e -containing tight, since we'd have $x(\text{dep}(x, e) \setminus \{e'\}) = x(\text{dep}(x, e)) = f(\text{dep}(x, e))$.

On Greedy, and linear programming max

Theorem 3.8

Let $y \in P_f$ be an extreme point, and let \preceq be the partial order of y . Let $c \in \mathbb{R}^E$. Then, y is the solution in:

$$c^T y = \max \{c^T x : x \in P_f\} \quad (7)$$

iff the following three conditions hold:

- (1) $c(e) \geq 0$ for every $e \in \text{supp}(y)$
- (2) $c(e) \leq 0$ for every $e \in E \setminus \text{sat}(y)$, and
- (3) For $d, e \in \text{sat}(y)$ and $d \preceq e$ imply that $c(d) \geq c(e)$.

Another revealing theorem

Theorem 3.9

Let f be a polymatroid function and suppose that E can be partitioned into (E_1, E_2, \dots, E_k) such that $f(A) = \sum_{i=1}^k f(A \cap E_i)$ for all $A \subseteq E$, and k is maximum. Then the base polytope $B_f = \{x \in P_f : x(E) = f(E)\}$ (the E -tight subset of P_f) has dimension $|E| - k$.

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- Thus, “independence” between disjoint A and B (leading to a rectangular projection of the polymatroid polytope) reduces the dimension of the base polytope, as expected.
- Thus, any point $x \in B_f$ is a convex combination of at most $|E| - k + 1$ vertices of B_f .

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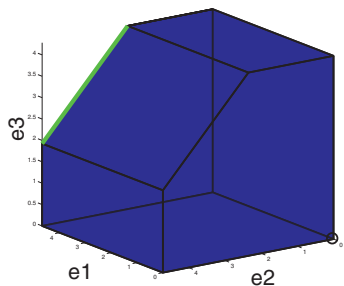
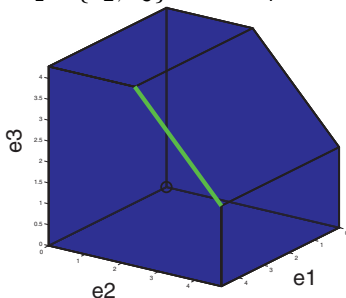
- Thus, “independence” between disjoint A and B (leading to a rectangular projection of the polymatroid polytope) reduces the dimension of the base polytope, as expected.
- Thus, any point $x \in B_f$ is a convex combination of at most $|E| - k + 1$ vertices of B_f .
- And if f does not have such independence, dimension of B_f is $|E| - 1$ and any point $x \in B_f$ is a convex combination of at most $|E|$ vertices of B_f .

Another revealing theorem

Theorem 3.9

Let f be a polymatroid function and suppose that E can be partitioned into (E_1, E_2, \dots, E_k) such that $f(A) = \sum_{i=1}^k f(A \cap E_i)$ for all $A \subseteq E$, and k is maximum. Then the base polytope $B_f = \{x \in P_f : x(E) = f(E)\}$ (the E -tight subset of P_f) has dimension $|E| - k$.

- Example f with independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$, with B_f marked in green.



Base polytope existence

- Given polymatroid function f , the base polytope $B_f = \{x \in \mathbb{R}_+^E : x(A) \leq f(A) \forall A \subseteq E, \text{ and } x(E) = f(E)\}$ always exists.

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- Consider **any** order of E and generate a vector x by this order (i.e., $x(e_1) = f(\{e_1\})$, $x(e_2) = f(\{e_1, e_2\}) - f(\{e_1\})$, and so on).

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 - $x \in P_f$
 - x is an extreme point in P_f
 - Since x is generated using an ordering of all of E , we have that $x(E) = f(E)$.
- Thus $x \in B_f$, and B_f is never empty.
- Moreover, in this case, x is a vertex of B_f since it is extremal.

Base polytope property

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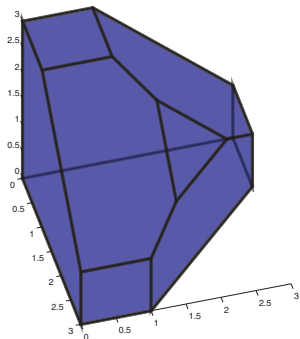
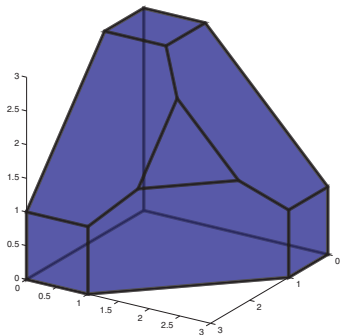
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- Then, we have generated a point x (a vertex, no less) in B_f such that $x(A) = f(A)$.
- Thus, for any A , we have

$$B_f \cap \left\{ x \in \mathbb{R}_+^E : x(A) = f(A) \right\} \neq \emptyset \quad (8)$$



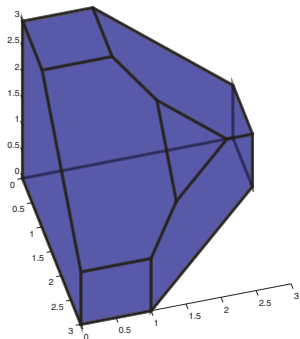
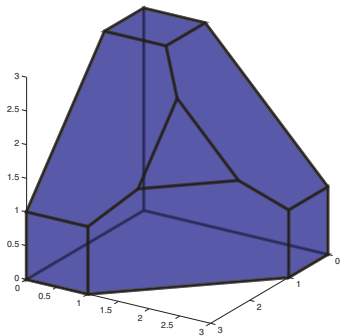
Polytope example 1

- Observe: P_f (at two views):



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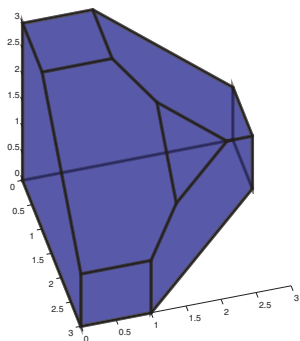
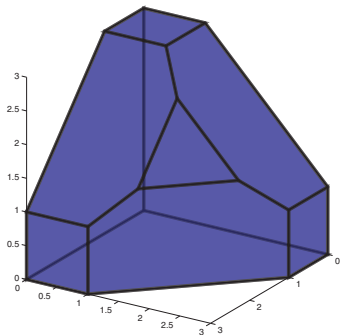
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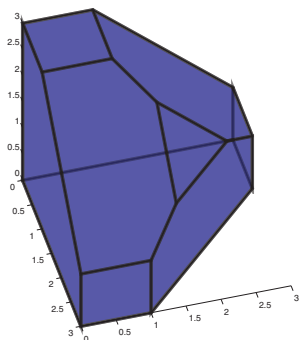
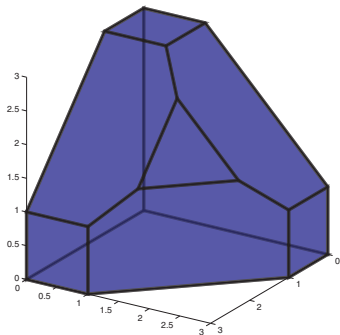
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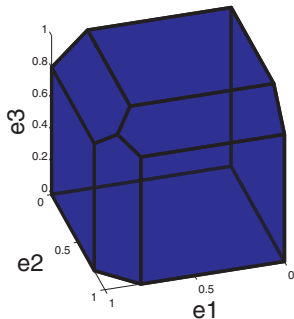
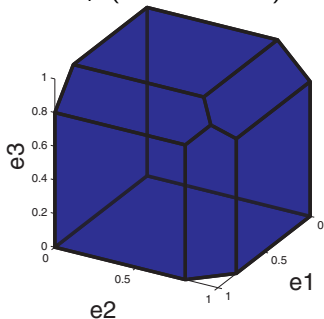
- Observe: P_f (at two views):



- Is this a polymatroidal polytope?
- No, " B_f " doesn't intersect sets of the form $\{x : x(e) = f(e)\}$.
- This was generated using function $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$.

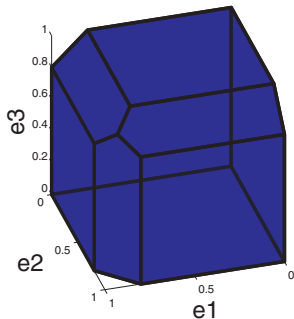
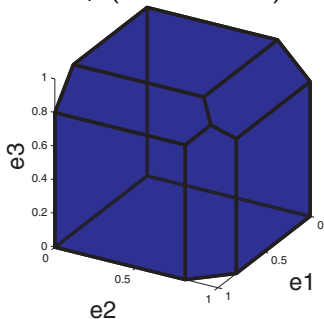
Polytope example 2

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Polytope example 2

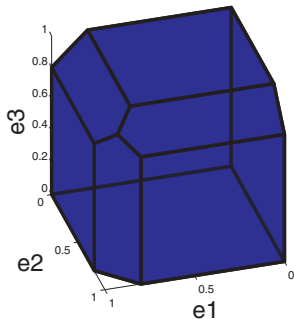
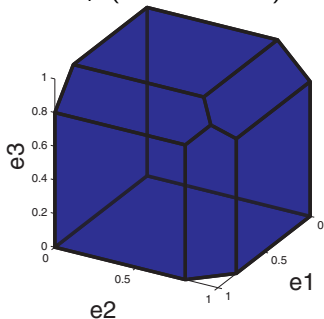
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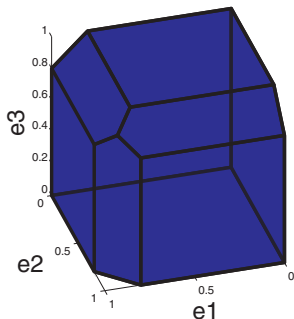
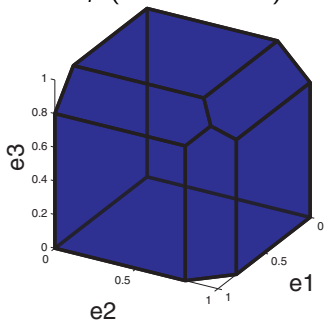


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- No, " B_f " doesn't intersect sets of the form $\{x : x(A) = t\}$ for $|A| < 2$.



Polytope example 2

- Observe: P_f (at two views):



- Is this a polymatroidal polytope?
- No, " B_f " doesn't intersect sets of the form $\{x : x(A) = f(A)\}$ for $|A| < 2$.
- This was generated using function $g(0) = 0$, $g(1) = 1$, $g(2) = 1.8$, and $g(3) = 3$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 1.8 + 1.8 = 3.6$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 3 + 1 = 4$.

→ SFM for arbitrary submodular g (from lecture 11)

- First, given **any** submodular function g , construct modular function $m : E \rightarrow \mathbb{R}$ such that $m(e) = g(E \setminus \{e\}) - g(E)$.

Note that

$$m(e) = g(E \setminus \{e\}) - g(E) \tag{9}$$

$$= -[g(E) - g(E \setminus e)] \tag{10}$$

$$= -[\textit{gain of adding } e \textit{ to } E \setminus e] \tag{11}$$

$$= -[\textit{smallest possible gain/value of } e \textit{ in any context}] \tag{12}$$

The last equality follows from submodularity.

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- Then $f(\emptyset) = 0$, so f is normalized.
- Also, f is monotone non-decreasing (and thus non-negative) and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since for $v \notin B$,

$$f(B + v) - f(B) = g(B + v) - g(B) + m(v) \quad (14)$$

$$= g(B + v) - g(B) + g(E - v) - g(E) \quad (15)$$

$$\geq 0 \quad (16)$$

since, by submodularity, $g(B + v) - g(B) \geq g(E) - g(E - v)$.

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Dealing with $m \in \mathbb{R}_+^E$

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- This follows since f is monotone non-decreasing, and $m(A) = m(A') + m(e)$, so $m(A') > m(A)$.
- So we “throw away” any e s.t. $m(e) < 0$. We get a new function on $E' = E \setminus M$ where $M = \{e : m(e) < 0\}$, and define new function $f' : 2^{E'} \rightarrow \mathbb{R}_+$ with $f'(A) = f(A)$ for $A \subseteq E'$. This deals with (2) above.

→ SFM on arbitrary submodular g : transformation

- Given any arbitrary submodular function g with the goal of finding $A^* \in \operatorname{argmin}_{A \subseteq E} g(A)$

- We reduce this to:

$$A^* \in \operatorname{argmin}_{A \subseteq E'} \left(f(A) - m(A) \right) \quad (17)$$

where

- f is a polymatroid function on $2^{E'}$
- m is a modular function on $2^{E'}$ with $m \in \mathbb{R}_+^{E'}$.
- $E' \subseteq E$.
- In the sequel, we assume this form, with ground set E .
- Moreover, we may assume that P_f is a polymatroidal polytope, with $P_f \subset \mathbb{R}_+^E$.



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Let f be a polymatroid function defined on subsets of E . For any $x \in \mathbb{R}_+^E$, then u

$$\max (y(E) : y \leq x, y \in P_f) = \min (f(A) + x(E \setminus A) : A \subseteq E) \quad (18)$$

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- foo

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- The nature of SFM will be very similar to the Edmonds's matroid partition problem (recall, asking if E can be partitioned into $\{I_i\}$ each independent in a matroid M_i) and the core algorithm is very similar.

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- That is, let I be an index set, and $x^{(i)}$ be an extreme point of P_f for $i \in I$. We then keep y as

$$y = \sum_{i \in I} \lambda_i x^{(i)} \quad (19)$$

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- Start with $y = 0$, $I = \{1\}$, $\lambda_1 = 1$, and $v^{(1)} = 0$.

From vertex to vertex

- We will need to move from one extreme point to another (adjacent) extreme point, and will use an augmenting path like approach to do so.
- How do we characterize such adjacent extreme points?

From vertex to vertex

Theorem 4.2

Let x be an extreme point of P_f , and let \preceq be its partial order. Then, each of the following three operations will yield a new extreme point w :

- (a) Let $a, b \in E$ and a cover b relative to \preceq . Let $w = x + \alpha \mathbf{1}_a - \alpha \mathbf{1}_b$ with $\alpha = f(\text{dep}(x, a) - b) - x(\text{dep}(x, a) - b)$.

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Theorem 4.2

Let x be an extreme point of P_f , and let \preceq be its partial order. Then, each of the following three operations will yield a new extreme point w :

- (a) Let $a, b \in E$ and a cover b relative to \preceq . Let $w = x + \alpha \mathbf{1}_a - \alpha \mathbf{1}_b$ with $\alpha = f(\text{dep}(x, a) - b) - x(\text{dep}(x, a) - b)$.
- (b) Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.

From vertex to vertex

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- (b) Let $a \in E \setminus \text{sat}(x)$, and let $w = x + \alpha \mathbf{1}_a$ where $\alpha = f(\text{sat}(x) + a) - f(\text{sat}(x))$.
- (c) Let $a \in \text{supp}(x)$ be maximal (w.r.t. \preceq), and let $w = x - x(a) \mathbf{1}_a$.

Scratch Paper

Scratch Paper

Scratch Paper

Sources for Today's Lecture

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- Lovász, "Submodular Functions and Convexity", 1983.