Announcements

• Homework 2 is due tonight at 11:45pm. All things in lectures marked “exercise”

• Again, all submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.

• Last lecture, all annotations apparently lost (unless you are a PDF expert). Please email me any typos you discover in lecture 14!!
We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): towards SFM
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?
Polymatroids

Theorem 2.1

For a given ordering \( E = (e_1, \ldots, e_m) \) of \( E \) and a given \( E_i \) and \( x \) generated by \( E_i \) using the greedy procedure, then \( x \) is an extreme point of \( P_f \).

Theorem 2.2

If \( x \) is an extreme point of \( P_f \) and \( B \subseteq E \) is given such that \( \text{supp}(x) \subseteq B \subseteq \text{sat}(x) \), then \( x \) is generated using greedy by some ordering of \( B \).
Partially ordered set

- A partially ordered set (poset) is a set of objects with an order.
Partially ordered set

- A partially ordered set (poset) is a set of objects with an order.
- In a poset, for any \( x, y, z \in V \) the following conditions hold (by definition):
  
  \[
  \begin{align*}
  &\text{For all } x, x \preceq x. \quad \text{(Reflexive)} \quad (P1.) \\
  &\text{If } x \preceq y \text{ and } y \preceq x, \text{ then } x = y \quad \text{(Antisymmetry)} \quad (P2.) \\
  &\text{If } x \preceq y \text{ and } y \preceq z, \text{ then } x \preceq z. \quad \text{(Transitivity)} \quad (P3.)
  \end{align*}
  \]
Partially ordered set

- A partially ordered set (poset) is a set of objects with an order.
- In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):
  
  For all $x, x \leq x$. (Reflexive) (P1.)
  
  If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry) (P2.)
  
  If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity) (P3.)

- The order $n(P)$ of a poset $P$ is meant the (cardinal) number of its elements.
Partially ordered set

Hasse-diagram: We can draw a poset using a graph where each \( x \in V \) is a node, and if \( x \sqsubseteq y \) we draw \( y \) directly above \( x \) with a connecting edge, but no other edges.

**Theorem 2.3**

Every non-empty finite subset \( X \subseteq V \) has a minimal (and maximal) element.
Partially ordered set

- Hasse-diagram: We can draw a poset using a graph where each \( x \in V \) is a node, and if \( x \sqsubseteq y \) we draw \( y \) directly above \( x \) with a connecting edge, but no other edges.
Hasse-diagram: We can draw a poset using a graph where each $x \in V$ is a node, and if $x \sqsubseteq y$ we draw $y$ directly above $x$ with a connecting edge, but no other edges.

**Theorem 2.3**

*Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.*
Theorem 2.4

Every non-empty finite subset \( X \subseteq V \) has a minimal (and maximal) element.

Proof.

Let \( X = \{x_1, \ldots, x_n\} \). Define \( m_1 = x_1 \) and

\[
m_k = \begin{cases} 
  x_k & \text{if } x_k \prec m_{k-1} \\
  m_{k-1} & \text{otherwise}
\end{cases}
\]

(1)

Then we have constructed \( m_n \preceq m_{n-1} \preceq \cdots \preceq m_1 \) meaning there is no \( m_k \) for \( k < n \) such that \( m_k \prec m_n \). Let \( M = \{m_1, \ldots, m_n\} \). By construction, we also have that there is no \( x \in X \) with \( x \prec m_n \), thus \( m_n \) is minimal.
Given a poset $V$, the length $\ell(V)$ is defined to be the l.u.b. of the lengths of any chains in $V$. That is, $\ell(V)$ is the least upper bound, i.e., smallest number not less than any chain length in $V$. 

The height or dimension of an element $x \in V$, or $h(x)$ is the l.u.b. of the lengths of the chains $0 = x_0 \prec x_1 \prec \ldots \prec x_l = x$ between $0$ and $x$. Note that $h(1) = \ell(V)$ when they exist. $h(x) = 1$ iff $0 \subseteq 1$ and such elements (with unit height) are called "atoms" or "points" or "(ground) elements".
Given a poset \( V \), the length \( \ell(V) \) is defined to be the l.u.b. of the lengths of any chains in \( V \). That is, \( \ell(V) \) is the least upper bound, i.e., smallest number not less than any chain length in \( V \).

The **height** or **dimension** of an element \( x \in V \), or \( l = h(x) \) is the l.u.b. of the lengths of the chains \( 0 = x_0 \prec x_1 \prec \ldots x_l = x \) between 0 and \( x \). Note that \( h(1) = \ell(V) \) when they exist. \( h(x) = 1 \) iff \( 0 \sqsubseteq x \) and such elements (with unit height) are called “atoms” or “points” or “(ground) elements”.

---

**Prof. Jeff Bilmes**

EE595A/Spr 2011/Submodular Functions – Lecture 15 - May 19th, 2011

---

**Page 8**
Definition 2.5 (Jordan-Dedekind Chain Condition)
(or JDCC) All maximal length chains between the same endpoints have the same finite length.
Definition 2.5 (Jordan-Dedekind Chain Condition)
(or JDCC) All maximal length chains between the same endpoints have the same finite length.

Theorem 2.6
Let $V$ be a poset with $0 \in V$ and where all chains are finite. Then $V$ satisfies JDCC iff it is graded by $h(x)$ (the height function).
Definition 2.5 (Jordan-Dedekind Chain Condition)
(or JDCC) All maximal length chains between the same endpoints have the same finite length.

Theorem 2.6
Let \( V \) be a poset with \( 0 \in V \) and where all chains are finite. Then \( V \) satisfies JDCC iff it is graded by \( h(x) \) (the height function).

- With JDCC, if \( x \sqsubseteq y \) then \( h(x) + 1 = h(y) \).
Partially ordered set

- With JDCC, element $x$ has height or rank $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between $x$ and $y$. 
Partially ordered set

- With JDCC, element $x$ has height or rank $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between $x$ and $y$.

- We say a poset is “graded” if it is graded by the height function.
Lattice defined

- Given $X \subset V$, $y \in V$ is an upper bound of $X$ if $x \leq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just sup$X$), then $y \leq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \leq y_2$ and $y_2 \leq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).
Lattice defined

- Given $X \subseteq V$, $y \in V$ is an upper bound of $X$ if $x \preceq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just sup$X$), then $y \preceq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \preceq y_2$ and $y_2 \preceq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).

Definition 2.7 (lattice)

A lattice is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or meet denoted by $x \wedge y \in V$, and also have a l.u.b. or join denoted by $x \vee y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).
Lattice defined

- Given $X \subset V$, $y \in V$ is an upper bound of $X$ if $x \leq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just sup$X$), then $y \leq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \leq y_2$ and $y_2 \leq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).

Definition 2.7 (lattice)

A lattice is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or meet denoted by $x \wedge y \in V$, and also have a l.u.b. or join denoted by $x \vee y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).

- Note again, that such l.u.b.’s and g.l.b.’s are unique if they exist.
Lattices

Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm\infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.
Lattices

• Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm \infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.

• All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \lor f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$. 
Lattices

- Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm\infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.

- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \lor f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$.

- Any non-empty lattice contains a greatest element $1 \in V$ and a least element $0 \in V$. 
Lattices

- Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm \infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.

- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \lor f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$.

- Any non-empty lattice contains a greatest element $1 \in V$ and a least element $0 \in V$.

- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.
Lattices

- Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm\infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.

- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \vee f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$.

- Any non-empty lattice contains a greatest element $1 \in V$ and a least element $0 \in V$.

- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.

- In a chain, $x \land y$ is the smaller of the two, and $x \lor y$ is the larger of the two.
Definition 2.8 (sublattice)

A sublattice of a lattice is a subset \( X \subseteq V \) such that join and meet are closed within \( X \) (for all \( x, y \in X \), \( x \vee y \in X \) and \( x \wedge y \in X \)). A sublattice is a lattice.
Definition 2.8 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.

- Given any $x \preceq y$, then all elements $\{z : x \preceq z \preceq y\}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.
Definition 2.8 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.

- Given any $x \preceq y$, then all elements $\{ z : x \preceq z \preceq y \}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.
Definition 2.8 (sublattice)

A sublattice of a lattice is a subset \( X \subseteq V \) such that join and meet are closed within \( X \) (for all \( x, y \in X \), \( x \lor y \in X \) and \( x \land y \in X \)). A sublattice is a lattice.

- Given any \( x \preceq y \), then all elements \( \{ z : x \preceq z \preceq y \} \) form a sublattice. We note that in such case, we say that \([x, y]\) form a (closed) interval in the lattice, and we have that the (closed) interval \([x, y]\) of all elements \( z \in L \) such that \( x \preceq z \preceq y \) is a sublattice.

- Obviously, \( 2^E \) for some set \( E \) is a lattice, with join/meet being union/intersection. See Figure(C).
Theorem 2.9

In any poset \( V \), the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

\[
\begin{align*}
\text{(Idempotent)} & \quad x \land x = x, \ x \lor x = x \\
\text{(Commutative)} & \quad x \land y = y \land x, \ x \lor y = y \lor x \\
\text{(Associative)} & \quad x \land (y \land z) = (x \land y) \land z, \ x \lor (y \lor z) = (x \lor y) \lor z \\
\text{(Absorption)} & \quad x \land (x \lor y) = x \lor (x \land y) = x \\
\text{(Consistency)} & \quad x \preceq y \iff x \land y = x \quad \text{and} \quad x \lor y = y
\end{align*}
\]
Theorem 2.9

In any poset \( V \), the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

\[
\begin{align*}
    x \land x &= x, \\
    x \lor x &= x \quad \text{(Idempotent)} \quad (L1) \\
    x \land y &= y \land x, x \lor y &= y \lor x \quad \text{(Commutative)} \quad (L2) \\
    x \land (y \land z) &= (x \land y) \land z, x \lor (y \lor z) &= (x \lor y) \lor z \quad \text{(Associative)} \quad (L3) \\
    x \land (x \lor y) &= x \lor (x \land y) &= x \quad \text{(Absorption)} \quad (L4) \\
    x \preceq y &\iff x \land y = x \quad \text{and} \ x \lor y = y \quad \text{(Consistency)} \quad (CON)
\end{align*}
\]

Note the above works for posets, not necessary for it to be a lattice.
Theorem 2.10

Given a poset \( V \) with \( 0 \in V \), then for all \( x \in V \),

\[ 0 \land x = 0 \text{ and } 0 \lor x = x \]  

(2)
Lattices

Theorem 2.10

*Given a poset* \( V \) *with* \( 0 \in V \), *then for all* \( x \in V \),

\[
0 \land x = 0 \quad \text{and} \quad 0 \lor x = x
\]

(2)

Theorem 2.11

*In any* lattice, the operations of join and meet are order-preserving in the following sense:

\[
y \leq z \Rightarrow x \land y \leq x \land z \quad \text{and} \quad x \lor y \leq x \lor z
\]

(3)
## Lattices

### Theorem 2.10

*Given a poset $V$ with $0 \in V$, then for all $x \in V$,*

$$0 \land x = 0 \text{ and } 0 \lor x = x \tag{2}$$

### Theorem 2.11

*In any lattice, the operations of join and meet are order-preserving in the following sense:*

$$y \preceq z \Rightarrow x \land y \preceq x \land z \text{ and } x \lor y \preceq x \lor z \tag{3}$$

### Theorem 2.12

*In any lattice, the following *distributive inequalities* hold for all $x, y, z \in V$:

$$x \land (y \lor z) \succeq (x \land y) \lor (x \land z) \tag{4a}$$

$$x \lor (y \land z) \preceq (x \lor y) \land (x \lor z) \tag{4b}$$
Distributive Inequalities

Theorem 2.13

In any lattice, the following distributive inequalities hold for all \( x, y, z \in V \):

1. \( x \land (y \lor z) \geq (x \land y) \lor (x \land z) \) (5a)
2. \( x \lor (y \land z) \leq (x \lor y) \land (x \lor z) \) (5b)
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

```
(D) 0
    a b c
        1
```

For example, in (D), we have that
\[ a \land (b \lor c) = a \land 1 = a \] but
\[ (a \land b) \lor (a \land c) = 0 \lor 0 = 0 \]
and obviously \( a \succ 0 \).

Also, in (D) we have
\[ a \lor (b \land c) = a \lor 0 = a \prec (a \lor b) \land (a \lor c) = 1 \land 1 = 1. \]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

For example, in (D), we have that $a \land (b \lor c) = a \land 1 = a$ but $(a \land b) \lor (a \land c) = 0 \lor 0 = 0$ and obviously $a \succ 0$. 

(D) $0$

1

a b c

(D) $0$

1

a b c

(D) $0$

1

a b c

(D) $0$
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

- For example, in (D), we have that \( a \land (b \lor c) = a \land 1 = a \) but \( (a \land b) \lor (a \land c) = 0 \lor 0 = 0 \) and obviously \( a \succ 0 \).

- Also, in (D) we have \( a \lor (b \land c) = a \lor 0 = a \vartriangleleft \)
  \( (a \lor b) \land (a \lor c) = 1 \land 1 = 1 \).
Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

In (E), we have that
\[ c \land (a \lor b) = c \land 1 = c \succ (c \land a) \lor (c \land b) = 0 \lor b = b \]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

\[
\begin{align*}
1 & \rightarrow \text{a} \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
0 & \rightarrow
\end{align*}
\]

- In (E), we have that
  \[
  c \land (a \lor b) = c \land 1 = c \succ (c \land a) \lor (c \land b) = 0 \lor b = b
  \]

- Also, in (E), we have
  \[
  b \lor (a \land c) \lor 0 = b \prec (b \lor a) \land (b \lor c) = 1 \land c = c
  \]
Modular inequality

Theorem 2.14

In any lattice, the following modular inequalities holds for all $x, y, z \in V$:

$$x \leq z \Rightarrow x \lor (y \land z) \leq (x \lor y) \land z$$

(6)
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

- Some lattices are such that the distributive inequality is an equality everywhere, and these are called distributive lattices. Only one quality is necessary since:

\[
\begin{align*}
\forall x, y, z \quad (x \land (y \lor z)) &= (x \land y) \lor (x \land z) \\
\forall x, y, z \quad (x \lor (y \land z)) &= (x \lor y) \land (x \lor z)
\end{align*}
\]
A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

Some lattices are such that the distributive inequality is an equality everywhere, and these are called distributive lattices. Only one quality is necessary since:

**Theorem 3.1**

*In any lattice, the following are equivalent:*

\[
\begin{align*}
 x \land (y \lor z) &= (x \land y) \lor (x \land z) \quad \forall x, y, z \\
 x \lor (y \land z) &= (x \lor y) \land (x \lor z) \quad \forall x, y, z
\end{align*}
\]

\[(7a)\]  
\[(7b)\]
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

- Some lattices are such that the distributive inequality is an equality everywhere, and these are called distributive lattices. Only one quality is necessary since:

**Theorem 3.1**

In any lattice, the following are equivalent:

\[
\begin{align*}
    x \land (y \lor z) &= (x \land y) \lor (x \land z) & \forall x, y, z \\
    x \lor (y \land z) &= (x \lor y) \land (x \lor z) & \forall x, y, z
\end{align*}
\]

(7a)  
(7b)

It is important to note the \( \forall x, y, z \) since this is not true only for individual elements. Note moreover that this means that the operators \( \lor = + \) and \( \land = \cdot \) do not form a lattice over \( \mathbb{R} \).
Theorem 3.2

In any lattice, the following are equivalent:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \quad (8a) \]

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \quad (8b) \]
Theorem 3.2

In any lattice, the following are equivalent:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \] (8a)

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \] (8b)

Proof.

Take as given the 2nd equation and show the first. Then

\[
(x \land y) \lor (x \land z) = [(x \land y) \lor x] \land [(x \land y) \lor z] \quad \text{by the 2nd eq}
\]

\[
= x \land [(x \land y) \lor z] \quad \text{(10)}
\]

\[
= x \land [(x \lor z) \land (y \lor z)] \quad \text{by the 2nd eq}
\]

\[
= x \land (x \lor z) \land (y \lor z) \quad \text{associative}
\]

\[
= x \land (y \lor z) \quad \text{(13)}
\]
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
- Thus a lattice is distributive if either of the above equalities hold.
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
- Thus a lattice is distributive if either of the above equalities hold.

Example 3.3

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \leq y$ mean that $x$ divides $y$. I.e., $2 \leq 4$ but $2 \not\leq 5$. Then this is lattice with $x \lor y = \text{l.c.m.}(x, y)$ and $x \land y = \text{g.c.d.}(x, y)$. It is also distributive.

Again consider figure (B).
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
- Thus a lattice is distributive if either of the above equalities hold.

**Example 3.3**

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that $x$ divides $y$. I.e., $2 \leq 4$ but $2 \not\leq 5$. Then this is lattice with $x \lor y = \text{l.c.m.}(x, y)$ and $x \land y = \text{g.c.d.}(x, y)$. It is also distributive.

Again consider figure (B).

**Theorem 3.4 (identity)**

*In a distributive lattice, if $z \land x = z \land y$ and $z \lor x = z \lor y$ then $x = y$.***
In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.
Modular Lattices

In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 4.1 (modular identity)**

\[ \forall x, y, z, \text{ if } x \preceq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \quad (L5) \]
Modular Lattices

In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 4.1 (modular identity)**

\[
\forall x, y, z, \text{ if } x \leq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \quad (L5)
\]

Clearly any distributive lattice satisfies the modular identity since when \( x \leq z \) we have that \( x \lor z = z \) and from the 2nd of the distributive lattice equalities (i.e., \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \)) we get the modular identity.
Modular Lattices

- In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 4.1 (modular identity)**

∀x, y, z, If x ≤ z, then x ∨ (y ∧ z) = (x ∨ y) ∧ z.  \(^{(L5)}\)

- Clearly any distributive lattice satisfies the modular identity since when x ≤ z we have that x ∨ z = z and from the 2nd of the distributive lattice equalities (i.e., x ∨ (y ∧ z) = (x ∨ y) ∧ (x ∨ z)) we get the modular identity.

- Easy way to remember. x, y, z and x ∨ (y ∧ z) = (x ∨ y) ∧ z
Modular Lattices

- In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

Definition 4.1 (modular identity)

\[ \forall x, y, z, \text{ if } x \leq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \quad (L5) \]

- Clearly any distributive lattice satisfies the modular identity since when \( x \leq z \) we have that \( x \lor z = z \) and from the 2nd of the distributive lattice equalities (i.e., \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \)) we get the modular identity.

- Easy way to remember. \( x, y, z \) and \( x \lor (y \land z) = (x \lor y) \land z \)
Modular Lattices

- In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 4.1 (modular identity)**

\[ \forall x, y, z, \text{ if } x \leq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \quad (L5) \]

- Clearly any distributive lattice satisfies the modular identity since when \( x \leq z \) we have that \( x \lor z = z \) and from the 2nd of the distributive lattice equalities (i.e., \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \)) we get the modular identity.

- Easy way to remember. \( x, y, z \) and \( x \lor (y \land z) = (x \lor y) \land z \)

- The term “modular” comes from abstract algebra, where a \( R \)-module is an abstract system that generalizes \( (\mathbb{R}, \mathbb{R}^n) \) (i.e., a vector field with scalar multiplication). An \( R \)-module ends up being a lattice that satisfies this identity.
Not every lattice is modular. Figure (D) is modular but not distributive. We already saw that (D) is not distributive since it is strict for certain assignments. It is modular though.
Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take \( b \preceq c \), then
\[
 b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land c = 1 \land c = c,
\]
so modular equality is violated.
Modular Lattices

- Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land c = 1 \land c = c$, so modular equality is violated.

**Theorem 4.2**

Any non-modular lattice $V$ contains the lattice in Figure (E) as a sublattice.
Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take \( b \preceq c \), then 
\[
 b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land c = 1 \land c = c ,
\]
so modular equality is violated.

**Theorem 4.2**

*Any non-modular lattice \( V \) contains the lattice in Figure (E) as a sublattice.*

Thus, the structure (E) is fundamental to non-modular lattices.
Modular Lattices

Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

Upper-Semimodularity if \( x \) and \( y \) cover \( z \) and \( x \neq y \) then \( x \lor y \) covers both \( x \) and \( y \), and

Lower-Semimodularity if \( z \) covers \( x \) and \( y \) and \( x \neq y \) then \( x \) and \( y \) both cover \( x \land y \).

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
Modular Lattices

Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if $x$ and $y$ cover $z$ and $x \neq y$ then $x \lor y$ covers both $x$ and $y$, and

Thus, upper-semimodularity means that if $z \sqsubseteq x$ and $z \sqsubseteq y$, and if $x \neq y$, then $x \sqsubseteq (x \lor y)$ and $y \sqsubseteq (x \lor y)$. 
Modular Lattices

Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if \( x \) and \( y \) cover \( z \) and \( x \neq y \), then \( x \lor y \) covers both \( x \) and \( y \), and

\[
\begin{align*}
\text{if } & x \sqsubseteq z \text{ and } y \sqsubseteq z, \text{ and } x \neq y, \text{ then } (x \land y) \sqsubseteq x \text{ and } (x \land y) \sqsubseteq y.
\end{align*}
\]

**Lower-Semimodularity** if \( z \) covers \( x \) and \( y \) and \( x \neq y \), then \( x \) and \( y \) both cover \( x \land y \).

Thus, lower-semimodularity means that if \( x \sqsupseteq z \) and \( y \sqsupseteq z \), and if \( x \neq y \), then \( (x \land y) \sqsupseteq x \) and \( (x \land y) \sqsupseteq y \).
Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if $x$ and $y$ cover $z$ and $x \neq y$ then $x \lor y$ covers both $x$ and $y$, and

**Lower-Semimodularity** if $z$ covers $x$ and $y$ and $x \neq y$ then $x$ and $y$ both covers $x \land y$.

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
Theorem 5.1

Let $L$ be a finite lattice. The following two conditions are equivalent:

(i) $L$ is graded, and the height function $h(\cdot)$ of $L$ satisfies the (what we know as the submodular) inequality for all $x, y \in L$.

$$h(x) + h(y) \geq h(x \lor y) + h(x \land y)$$  \hspace{1cm} (14)

(ii) If $x$ and $y$ both cover $z$, then $x \lor y$ covers both $x$ and $y$

\[ \iff \text{upper semimodular}. \]
Semi-modular/Submodular Lattices: (i) $\Rightarrow$ (ii)

Suppose $x$ and $y$ cover $z$.

$h$ submodular $\Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\}$. 

...
Semi-modular/Submodular Lattices: (i) ⇒ (ii)

\[ h \text{ submodular} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} . \]

- Suppose \( x \) and \( y \) cover \( z \).
- Note that if \( x \) and \( y \) cover \( z \) then since \( L \) is a lattice, \( z = x \land y \).
Semi-modular/Submodular Lattices: \((i) \Rightarrow (ii)\)

\[ h \text{ submodular} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\}. \]

- Suppose \(x\) and \(y\) cover \(z\).
- Note that if \(x\) and \(y\) cover \(z\) then since \(L\) is a lattice, \(z = x \land y\).
- Then we have \(h(x) = h(y) = h(x \land y) + 1\).
**Semi-modular/Submodular Lattices: (i) ⇒ (ii)**

<table>
<thead>
<tr>
<th>$h$ submodular</th>
<th>$\Rightarrow \left{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right}$</th>
</tr>
</thead>
</table>

- Suppose $x$ and $y$ cover $z$.
- Note that if $x$ and $y$ cover $z$ then since $L$ is a lattice, $z = x \land y$.
- Then we have $h(x) = h(y) = h(x \land y) + 1$.
- Also, since $x$ and $y$ are distinct, and since they both cover $z$ we can’t have (w.l.o.g.) $x \preceq y$, and thus $h(x \lor y) > h(x) = h(y)$. 
Semi-modular/Submodular Lattices: (i) \(\Rightarrow\) (ii)

\[ h_{\text{submodular}} \Rightarrow \{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \]

- Suppose \(x\) and \(y\) cover \(z\).
- Note that if \(x\) and \(y\) cover \(z\) then since \(L\) is a lattice, \(z = x \land y\).
- Then we have \(h(x) = h(y) = h(x \land y) + 1\).
- Also, since \(x\) and \(y\) are distinct, and since they both cover \(z\) we can’t have (w.l.o.g.) \(x \preceq y\), and thus \(h(x \lor y) > h(x) = h(y)\).
- Hence by (i), we have

\[ h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \]  \(15\)

...
Semi-modular/Submodular Lattices: (i) \( \Rightarrow \) (ii)

\[ h_{\text{submodular}} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \]

- Suppose \( x \) and \( y \) cover \( z \).
- Note that if \( x \) and \( y \) cover \( z \) then since \( L \) is a lattice, \( z = x \land y \).
- Then we have \( h(x) = h(y) = h(x \land y) + 1 \).
- Also, since \( x \) and \( y \) are distinct, and since they both cover \( z \) we can’t have (w.l.o.g.) \( x \preceq y \), and thus \( h(x \lor y) > h(x) = h(y) \).
- Hence by (i), we have
  \[ h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \quad (15) \]
- or
  \[ h(x \land y) + 2 \geq h(x \lor y) > h(x \land y) + 1 \quad (16) \]

...
Semi-modular/Submodular Lattices: $\text{(i)} \Rightarrow \text{(ii)}$

$h_{\text{submodular}} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\}$.

- Suppose $x$ and $y$ cover $z$.
- Note that if $x$ and $y$ cover $z$ then since $L$ is a lattice, $z = x \land y$.
- Then we have $h(x) = h(y) = h(x \land y) + 1$.
- Also, since $x$ and $y$ are distinct, and since they both cover $z$ we can’t have (w.l.o.g.) $x \preceq y$, and thus $h(x \lor y) > h(x) = h(y)$.
- Hence by (i), we have
  $$h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \quad (15)$$
- or
  $$h(x \land y) + 2 \geq h(x \lor y) > h(x \land y) + 1 \quad (16)$$
- giving $h(x \lor y) = h(x \land y) + 2 = h(x) + 1 = h(y) + 1$, so that $x \lor y$ covers both $x$ and $y$. 

...
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Suppose \(L\) is not graded, and let \([u, v]\) be an interval of \(L\) of minimal length that is not graded (so all smaller length intervals are graded).
Semi-modular/Submodular Lattices: \( (ii) \Rightarrow (i) \)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular}.
\]

- Suppose \( L \) is not graded, and let \([u, v]\) be an interval of \( L \) of minimal length that is not graded (so all smaller length intervals are graded).

- Then there are elements \( x_1, x_2 \) of \([u, v]\) where each of \( x_1 \) and \( x_2 \) cover \( u \),
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
\]

- Suppose \(L\) is not graded, and let \([u, v]\) be an interval of \(L\) of minimal length that is not graded (so all smaller length intervals are graded).
- Then there are elements \(x_1, x_2\) of \([u, v]\) where each of \(x_1\) and \(x_2\) cover \(u\),
- By the minimality, all maximal chains of each interval \([x_i, v]\) have the same length \(\ell_i\) where \(\ell_1 \neq \ell_2\).
Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

\[
\left\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\right\} \Rightarrow h \text{ submodular.}
\]

- Suppose $L$ is not graded, and let $[u, v]$ be an interval of $L$ of minimal length that is not graded (so all smaller length intervals are graded).
- Then there are elements $x_1, x_2$ of $[u, v]$ where each of $x_1$ and $x_2$ cover $u$.
- By the minimality, all maximal chains of each interval $[x_i, v]$ have the same length $\ell_i$ where $\ell_1 \neq \ell_2$.
- By (ii), there are saturated chains in $[x_i, v]$ of the form $x_i \prec x_1 \lor x_2 \prec y_1 \prec y_2 \prec \cdots \prec y_k = v$, contradicting $\ell_1 \neq \ell_2$. Hence $L$ is graded (i.e., every maximal chain has the same length, i.e., JDCC holds).
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Suppose \(L\) is not graded, and let \([u, v]\) be an interval of \(L\) of minimal length that is not graded (so all smaller length intervals are graded).
- Then there are elements \(x_1, x_2\) of \([u, v]\) where each of \(x_1\) and \(x_2\) cover \(u\),
- By the minimality, all maximal chains of each interval \([x_i, v]\) have the same length \(\ell_i\) where \(\ell_1 \neq \ell_2\).
- By \((ii)\), there are saturated chains in \([x_i, v]\) of the form \(x_i \prec x_1 \lor x_2 \prec y_1 \prec y_2 \prec \cdots \prec y_k = v\), contradicting \(\ell_1 \neq \ell_2\).
- Hence \(L\) is graded (i.e., every maximal chain has the same length, i.e., JDCC holds).
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \vee y)), (y \sqsubseteq (x \vee y))\} \Rightarrow h \text{ submodular.}
\]

- Now suppose there is a pair \(x, y \in L\) violating the submodularity inequality, i.e., with
  \[
h(x) + h(y) < h(x \vee y) + h(x \wedge y)
  \] (17)
  and choose such a pair first with \(\ell(x \wedge y, x \vee y)\) minimal, and then (second) with \(h(x) + h(y)\) minimal.
Semi-modular/Submodular Lattices: (ii) \( \Rightarrow \) (i)

\[
\left\{ (z \sqcap x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
\]

- Now suppose there is a pair \( x, y \in L \) violating the submodularity inequality, i.e., with

\[
h(x) + h(y) < h(x \lor y) + h(x \land y)
\]

and choose such a pair first with \( \ell(x \land y, x \lor y) \) minimal, and then (second) with \( h(x) + h(y) \) minimal.

- By (ii), we cannot have both \( x \) and \( y \) covering \( x \land y \) (because if we did, then \( h(x) = h(x \land y) + 1 \), \( h(y) = h(x \land y) + 1 \), and (ii) gives that \( h(x \lor y) = h(x) + 1 = h(y) + 1 \), and we would have the submodular inequality at equality).
Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

\[
\left\{(z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \lor y)), (y \sqsubset (x \lor y))\right\} \Rightarrow h \text{ submodular.}
\]

- Now suppose there is a pair $x, y \in L$ violating the submodularity inequality, i.e., with

\[
h(x) + h(y) < h(x \lor y) + h(x \land y)
\]  

and choose such a pair first with $\ell(x \land y, x \lor y)$ minimal, and then (second) with $h(x) + h(y)$ minimal.

- By (ii), we cannot have both $x$ and $y$ covering $x \land y$ (because if we did, then $h(x) = h(x \land y) + 1$, $h(y) = h(x \land y) + 1$, and (ii) gives that $h(x \lor y) = h(x) + 1 = h(y) + 1$, and we would have the submodular inequality at equality).

- Thus assume that $x \land y \prec x' \prec x$, say (w.l.o.g.)

\[\begin{array}{c}
y \\
y' \\
x \land y \\
x'
\end{array}\]
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Now suppose there is a pair \(x, y \in L\) violating the submodularity inequality, i.e., with

\[
h(x) + h(y) < h(x \lor y) + h(x \land y)
\]

and choose such a pair first with \(\ell(x \land y, x \lor y)\) minimal, and then (second) with \(h(x) + h(y)\) minimal.

- By \((ii)\), we cannot have both \(x\) and \(y\) covering \(x \land y\) (because if we did, then \(h(x) = h(x \land y) + 1\), \(h(y) = h(x \land y) + 1\), and \((ii)\) gives that \(h(x \lor y) = h(x) + 1 = h(y) + 1\), and we would have the submodular inequality at equality).

- Thus assume that \(x \land y \prec x' \prec x\), say (w.l.o.g.)

- By the minimality of \(\ell(x \land y, x \lor y)\) and \(h(x) + h(y)\), we have

\[
h(x') + h(y) \geq h(x' \land y) + h(x' \lor y).
\]
Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

$$\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}$$

Now $x' \land y = x \land y$, so Eq. 17 and Eq. 18 together imply that

$$h(x) + h(x' \lor y) < h(x') + h(x \lor y).$$  \hspace{1cm} (19)
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
\]

- Now \(x' \land y = x \land y\), so Eq. 17 and Eq. 18 together imply that
  \[
  h(x) + h(x' \lor y) < h(x') + h(x \lor y).
  \] (19)
- Since \(x \succ x'\), we have \(x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y\).
{(z ⊑ x, z ⊑ y) ⇒ (x ⊑ (x ∨ y)), (y ⊑ (x ∨ y))} ⇒ h submodular.

Now x' ∧ y = x ∧ y, so Eq. 17 and Eq. 18 together imply that

\[ h(x) + h(x' ∨ y) < h(x') + h(x ∨ y). \]  (19)

Since x ⪰ x', we have x ∨ (x' ∨ y) = (x ∨ x') ∨ y = x ∨ y.

Also, by the modular inequalities (with x ← x', y ← y, z ← x), we have x ∧ (x' ∨ y) ⪰ x' ∨ (y ∧ x) ⪰ x'.
Semi-modular/Submodular Lattices: \((\text{ii}) \Rightarrow (\text{i})\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \vee y)), (y \sqsubseteq (x \vee y))\} \Rightarrow h \text{ submodular.}
\]

\begin{itemize}
  \item Now \(x' \land y = x \land y\), so Eq. 17 and Eq. 18 together imply that
  \[
h(x) + h(x' \lor y) < h(x') + h(x \lor y).
  \]
  \[(19)\]
  \item Since \(x \succ x'\), we have \(x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y\).
  \item Also, by the modular inequalities (with \(x \leftarrow x', y \leftarrow y, z \leftarrow x\)), we have \(x \land (x' \lor y) \geq x' \lor (y \land x) \geq x'\).
  \item Hence setting \(X = x, Y = x' \lor y\). This gives \(X \lor Y = x \lor y\), and \(X \land Y \succeq x' \succ x\).
\end{itemize}
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}

- Now $x' \land y = x \land y$, so Eq. 17 and Eq. 18 together imply that
  \[ h(x) + h(x' \lor y) < h(x') + h(x \lor y). \]  
  (19)

- Since $x \succ x'$, we have $x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y$.

- Also, by the modular inequalities (with $x \leftarrow x'$, $y \leftarrow y$, $z \leftarrow x$), we have $x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x'$.

- Hence setting $X = x$, $Y = x' \lor y$. This gives $X \lor Y = x \lor y$, and $X \land Y \succeq x' \succ x$.

- Thus, we have found a pair $X, Y \in L$ with
  \[ h(X) + h(Y) < h(X \land Y) + h(X \lor Y) \]  and a strictly shorter length
  \[ \ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y), \]
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
\]

- Now \(x' \land y = x \land y\), so Eq. 17 and Eq. 18 together imply that
  \[
  h(x) + h(x' \lor y) < h(x') + h(x \lor y).
  \]  \hspace{1cm}(19)
- Since \(x \succ x'\), we have \(x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y\).
- Also, by the modular inequalities (with \(x \leftarrow x', y \leftarrow y, z \leftarrow x\)), we have \(x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x'\).
- Hence setting \(X = x, Y = x' \lor y\). This gives \(X \lor Y = x \lor y\), and \(X \land Y \succeq x' \succ x\).
- Thus, we have found a pair \(X, Y \in L\) with
  \[
  h(X) + h(Y) < h(X \land Y) + h(X \lor Y)
  \]
  and a strictly shorter length
  \[
  \ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y),
  \]
- This contradicts the minimality of \(\ell(x \land y, x \lor y)\).
Semi-modular/Submodular Lattices: (ii) \( \Rightarrow \) (i)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
\]

- Now \( x' \land y = x \land y \), so Eq. 17 and Eq. 18 together imply that
  \[
h(x) + h(x' \lor y) < h(x') + h(x \lor y).
  \]
  (19)
- Since \( x \succ x' \), we have \( x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y \).
- Also, by the modular inequalities (with \( x \leftarrow x', y \leftarrow y, z \leftarrow x \)), we have \( x \land (x' \lor y) \geq x' \lor (y \land x) \geq x' \).
- Hence setting \( X = x, Y = x' \lor y \). This gives \( X \lor Y = x \lor y \), and \( X \land Y \geq x' \succ x \).
- Thus, we have found a pair \( X, Y \in L \) with
  \[
h(X) + h(Y) < h(X \land Y) + h(X \lor Y)
  \]
  and a strictly shorter length
  \[
  \ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y),
  \]
- This contradicts the minimality of \( \ell(x \land y, x \lor y) \).
- The proof is complete.
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

```
  x \lor y
  \downarrow
   \cdot
     \uparrow
    x
  \downarrow
   \cdot
     \uparrow
    y
  \downarrow
   \cdot
     \uparrow
    x \land y

height
3
2
1
0

submodularity
h(x)+h(y)
> h(x \lor y)
+ h(x \land y)

2 + 2 > 3 + 0
```

- This lattice is not modular since $x \lor y$ covers $x$ and $y$, but $x$ and $y$ don’t cover $x \land y$. 

Flip it up side down to get a lower-semimodular (or “supermodular”) lattice.
Submodular Lattices

The next figure is an example of an upper-semimodular (or a "submodular") lattice over 7 elements.

\[ z = x \lor y \]

Also notes, this violates the modular equality
\[
(\forall x, y, z, \quad x \leq z \Rightarrow (x \lor (y \land z)) = (x \lor y) \land z).
\]

\[ \forall \leq \]

\[ \forall \]

\[ \exists \]

\[ \forall \leq \]

\[ \forall \leq \]
The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

Also notes, this violates the modular equality

\[(\forall x, y, z, \ x \leq z \Rightarrow (x \lor (y \land z) = (x \lor y) \land z)).\]

Flip it up side down to get a lower-semimodular (or “supermodular”) lattice.
Ideal in a Lattice

Definition 6.1 (ideal)

An ideal is a nonvoid subset $J$ of a lattice $L$ with the properties

$$\forall a \in J, x \in L, \ x \leq a \Rightarrow x \in J$$

(20)

$$\forall a \in J, \ b \in J \Rightarrow a \lor b \in J.$$  

(21)

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).
Ideal in a Lattice

**Definition 6.1 (ideal)**

An ideal is a nonvoid subset $J$ of a lattice $L$ with the properties

$$\forall a \in J, x \in L, \ x \leq a \Rightarrow x \in J$$

$$\forall a \in J, \ b \in J \Rightarrow a \lor b \in J.$$  \hspace{1cm} (20)

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).

**Proposition 6.2**

$J$ is an ideal when $a \lor b \in J$ iff $a \in J, b \in J$ (closure under join).
Definition 6.1 (ideal)

An ideal is a nonvoid subset $J$ of a lattice $L$ with the properties

$$\forall a \in J, x \in L, \ x \leq a \Rightarrow x \in J$$  \quad (20)

$$\forall a \in J, \ b \in J \Rightarrow a \lor b \in J.$$  \quad (21)

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).

Proposition 6.2

$J$ is an ideal when $a \lor b \in J$ iff $a \in J$, $b \in J$ (closure under join).

Example 6.3

In $2^E$, take any $A \subseteq E$, then $L(A) = \{B : B \subseteq A\}$ is an ideal in a set lattice.
Ideal in a Lattice

**Definition 6.4**

Given an element $a \in L$ in a lattice, the set $L(a)$ of all elements $\{x : x \preceq a, x \in L\}$ is an ideal, and is called a principle ideal.
**Ideal in a Lattice**

**Definition 6.4**

Given an element $a \in L$ in a lattice, the set $L(a)$ of all elements $\{x : x \preceq a, x \in L\}$ is an ideal, and is called a **principle ideal**.

In fact, in any finite lattice, every (nonvoid) ideal is a principle ideal. In fact, we have:
Definition 6.4

Given an element \( a \in L \) in a lattice, the set \( L(a) \) of all elements \( \{x : x \preceq a, x \in L\} \) is an ideal, and is called a principle ideal.

In fact, in any finite lattice, every (nonvoid) ideal is a principle ideal. In fact, we have:

Theorem 6.5

The set of all ideals of any lattice \( L \), ordered by inclusion, itself forms a lattice. The set of all principal ideals in \( L \) forms a sublattice of this lattice, which is isomorphic with \( L \).
Ideal in a Lattice

Definition 6.4

Given an element \( a \in L \) in a lattice, the set \( L(a) \) of all elements \( \{ x : x \preceq a, x \in L \} \) is an ideal, and is called a principle ideal.

In fact, in any finite lattice, every (nonvoid) ideal is a principle ideal. In fact, we have:

Theorem 6.5

The set of all ideals’of any lattice \( L \), ordered by inclusion, itself forms a lattice. The set of all principal ideals in \( L \) forms a sublattice of this lattice, which is isomorphic with \( L \).

Example 6.6

Consider \( 2^E \). Then for any \( A \subseteq E \), we see that \( L(A) = \{ B : B \subseteq A \} \) is an ideal. Also, we can see that the set of sets \( \{ L(A) : A \subseteq E \} \) is isomorphic to \( 2^E \) and also forms a lattice.
Complement and Complemented Lattices

Definition 6.7

A lattice with a 0 and 1 is complemented if for all \( x \in L \) there exists a \( y \in L \) such that \( x \lor y = 1 \) and \( x \land y = 0 \).
Definition 6.7

A lattice with a 0 and 1 is complemented if for all \( x \in L \) there exists a \( y \in L \) such that \( x \lor y = 1 \) and \( x \land y = 0 \). A lattice is relatively complemented if every interval \([x, y]\) is complemented (w.r.t. the interval, with \( x \) taking the role of 0 and \( y \) taking the role of 1).
Complement and Complemented Lattices

**Definition 6.7**

A lattice with a 0 and 1 is **complemented** if for all $x \in L$ there exists a $y \in L$ such that $x \lor y = 1$ and $x \land y = 0$. A lattice is **relatively complemented** if every interval $[x, y]$ is complemented (w.r.t. the interval, with $x$ taking the role of 0 and $y$ taking the role of 1).

Recall, an atom of a finite lattice is an element covering 0.
Complement and Complemented Lattices

Definition 6.7
A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \lor y = 1$ and $x \land y = 0$. A lattice is relatively complemented if every interval $[x, y]$ is complemented (w.r.t. the interval, with $x$ taking the role of 0 and $y$ taking the role of 1).

Recall, an atom of a finite lattice is an element covering 0.

Proposition 6.8
Any complemented modular lattice is relatively complemented.
Complement and Complemented Lattices

Definition 6.7

A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \lor y = 1$ and $x \land y = 0$. A lattice is relatively complemented if every interval $[x, y]$ is complemented (w.r.t. the interval, with $x$ taking the role of 0 and $y$ taking the role of 1).

Recall, an atom of a finite lattice is an element covering 0.

Proposition 6.8

Any complemented modular lattice is relatively complemented.

Proposition 6.9

In a complemented modular lattice of finite length, every element is the join of those elements which it contains.
Definition 6.10

A **boolean lattice** is a complemented distributive lattice.
Boolean Lattices

Definition 6.10

A **boolean lattice** is a complemented distributive lattice.

Theorem 6.11

_In any boolean lattice, each element $x$ has a unique complement $x'$. Moreover, we have_

\[
    x \land x' = 0, \quad x \lor x' = 1 \quad \text{(L1)}
\]

\[
    (x')' = x \quad \text{(L2)}
\]

\[
    (x \land y)' = x' \lor y', \quad (x \lor y)' = x' \land y' \quad \text{(L3)}
\]

Prof. Jeff Bilmes
Definition 6.12

An element $x$ of a lattice is called **join irreducible** if $y \lor z = x$ implies $y = x$ or $z = x$ (i.e., if $x$ is the join of two elements, it must be one of those elements).
**Definition 6.12**

An element $x$ of a lattice is called **join irreducible** if $y \lor z = x$ implies $y = x$ or $z = x$ (i.e., if $x$ is the join of two elements, it must be one of those elements). Equivalently, an element $x$ of a lattice is join irreducible if one cannot write $x = y \lor z$ where $y \prec x$ and $z \prec x$ (i.e., $x$ is not the join of two strictly smaller elements).
Join Irreducible

Definition 6.12

An element $x$ of a lattice is called **join irreducible** if $y \lor z = x$ implies $y = x$ or $z = x$ (i.e., if $x$ is the join of two elements, it must be one of those elements). Equivalently, an element $x$ of a lattice is join irreducible if one cannot write $x = y \lor z$ where $y \prec x$ and $z \prec x$ (i.e., $x$ is not the join of two strictly smaller elements).

Proposition 6.13

*If all chains in a lattice are finite, then every $a \in L$ can be represented as a join $a = x_1 \lor \ldots \lor x_n$ of a finite number of join irreducible elements.*
Join Irreducible

Definition 6.12
An element \( x \) of a lattice is called join irreducible if \( y \lor z = x \) implies \( y = x \) or \( z = x \) (ie, if \( x \) is the join of two elements, it must be one of those elements). Equivalently, an element \( x \) of a lattice is join irreducible if one cannot write \( x = y \lor z \) where \( y \prec x \) and \( z \prec x \) (ie, \( x \) is not the join of two strictly smaller elements).

Proposition 6.13
If all chains in a lattice are finite, then every \( a \in L \) can be represented as a join \( a = x_1 \lor \ldots \lor x_n \) of a finite number of join irreducible elements.

Proposition 6.14
In any complemented modular lattice, all join irreducible elements are atoms.
Definition 6.15 (ring family).

A ring of sets is a family $\Phi$ of subsets of a set $E$ which contains with any two sets $S$ and $T$ also their (set-theoretic) intersection $S \cap T$ and union $S \cup T$. A field of sets is a ring of sets which contains with any $S$ also its set complement $E \setminus S'$. 
A ring of sets is a family $\Phi$ of subsets of a set $E$ which contains with any two sets $S$ and $T$ also their (set-theoretic) intersection $S \cup T$ and union $S \cap T$. A field of sets is a ring of sets which contains with any $S$ also its set complement $E \setminus S$.

Thus, any ring of sets under the natural ordering $S \subset T$ forms a distributive lattice.
Join irredicible, ground elements, Boolean lattices

**Theorem 6.16**

*Let L be any distributive lattice of length n. Then the poset X of join-irreducible elements x ≳ 0 has order n and, moreover, L ≌ 2^X.*
Theorem 6.16

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L \cong 2^X$

- This means that any distributive lattice is the power set of some underlying ground set.
Join irreducible, ground elements, Boolean lattices

Theorem 6.16

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L = 2^X$.

- This means that any distributive lattice is the power set of some underlying ground set.
- The join-irreducible elements of a distributive lattice constitute the “ground elements” which generate the distributive lattice.
Join irreducible, ground elements, Boolean lattices

Theorem 6.16

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L = 2^X$

- This means that any distributive lattice is the power set of some underlying ground set.
- The join-irreducible elements of a distributive lattice constitute the “ground elements” which generate the distributive lattice.
- Thus, any distributive lattice of length $n$ is isomorphic with a ring of subsets of a set $E$ of $n$ elements.
Join irreducible, ground elements, Boolean lattices

**Theorem 6.16**

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L = 2^X$.

- This means that any distributive lattice is the power set of some underlying ground set.
- The join-irreducible elements of a distributive lattice constitute the “ground elements” which generate the distributive lattice.
- Thus, any distributive lattice of length $n$ is isomorphic with a ring of subsets of a set $E$ of $n$ elements.
- The next result is perhaps not so surprising.
Theorem 6.16

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L = 2^X$.

- This means that any distributive lattice is the power set of some underlying ground set.
- The join-irreducible elements of a distributive lattice constitute the “ground elements” which generate the distributive lattice.
- Thus, any distributive lattice of length $n$ is isomorphic with a ring of subsets of a set $E$ of $n$ elements.
- The next result is perhaps not so surprising.

Theorem 6.17

Every Boolean lattice of finite length $n$ is isomorphic with the field of all subsets of a set of $|E| = n$ elements, namely $2^E$. 
supp, sat, and dep

For $x \in P_f$, supp($x$) = \{e : x(e) \neq 0\}
For $x \in P_f$, $\text{supp}(x) = \{ e : x(e) \neq 0 \}$

For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e.,

$\text{sat}(x) = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$. That is,

$$\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x) \}$$ (22)

$$= \bigcup \{ A : A \subseteq E, x(A) = f(A) \}$$ (23)
supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{ e : x(e) \neq 0 \}$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e., $\text{sat}(x) = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$. That is,

$$\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x) \}$$

$$\quad = \bigcup \{ A : A \subseteq E, x(A) = f(A) \}$$

$$\quad = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$$
supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{ e : x(e) \neq 0 \}$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e.,
  \[ \text{sat}(x) = \{ e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f \} \]. That is,
  \[
  \text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x) \} \quad (22)
  = \bigcup \{ A : A \subseteq E, x(A) = f(A) \} \quad (23)
  = \{ e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f \} \quad (24)
  \]

- For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated ($x$-tight) set w.r.t. $x$ containing $e$. That is,
  \[
  \text{dep}(x, e) = \begin{cases} 
  \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\
  \emptyset & \text{else}
  \end{cases}
  \]
Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( \mathcal{D}(x) = \{ A : x(A) = f(A) \} \).
supp, sat, and dep

- Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( \mathcal{D}(x) = \{ A : x(A) = f(A) \} \).

- supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).
Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( D(x) = \{A : x(A) = f(A)\} \).

supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).

For \( x \in P_f \), with \( x \) extremal, is supp(x) = sat(x)?
supp, sat, and dep

- Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( \mathcal{D}(x) = \{ A : x(A) = f(A) \} \).

- supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).

- For \( x \in P_f \), with \( x \) extremal, is supp(\( x \)) = sat(\( x \))?

- Consider the case where disjoint \( A, B \subseteq E \), we have \( f(A) = f(B) = f(A \cup B) \) (meaning perfect dependence / redundancy).

\[
\begin{align*}
\text{f}(A \cup B) &= \text{f}(A) \lor \text{f}(B)
\end{align*}
\]
supp, sat, and dep

- Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( D(x) = \{ A : x(A) = f(A) \} \).
- supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).
- For \( x \in P_f \), with \( x \) extremal, is \( \text{supp}(x) = \text{sat}(x) \)?
- Consider the case where disjoint \( A, B \subseteq E \), we have \( f(A) = f(B) = f(A \cup B) \) (meaning perfect dependence / redundancy).
- Suppose \( x \in P_f \) has \( x(A) > 0 \) but \( x(B) = 0 \).
  \[
  x(A \cup B) = x(A) + x(B) = x(A)
  \]
Logistics
Review
Distributive Lattices
Modular Lattices
Submodular Lattices
Lattice
Extreme
Scratch
Summary

supp, sat, and dep

- Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice $\mathcal{D}(x) = \{A : x(A) = f(A)\}$.
- supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by $x(E_i) = f(E_i)$).
- For $x \in P_f$, with $x$ extremal, is supp(x) = sat(x)?
- Consider the case where disjoint $A, B \subseteq E$, we have $f(A) = f(B) = f(A \cup B)$ (meaning perfect dependence / redundancy).
- Suppose $x \in P_f$ has $x(A) > 0$ but $x(B) = 0$.
- Then supp(x) = A
Now, \( \text{sat}(x) \) is tight, and corresponds to the largest member of the distributive lattice \( D(x) = \{ A : x(A) = f(A) \} \).

\( \text{supp}(x) \) is not nec. tight, but for an extremal point, \( \text{supp}(x) \) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).

For \( x \in P_f \), with \( x \) extremal, is \( \text{supp}(x) = \text{sat}(x) \)?

Consider the case where disjoint \( A, B \subseteq E \), we have \( f(A) = f(B) = f(A \cup B) \) (meaning perfect dependence / redundancy).

Suppose \( x \in P_f \) has \( x(A) > 0 \) but \( x(B) = 0 \).

Then \( \text{supp}(x) = A \)

\( \text{sat}(x) = \bigcup \{ A : x(A) = f(A) \} \) and since \( x(A \cup B) = x(A) = f(A) = f(A \cup B) \), here, \( \text{sat}(x) = A \cup B \).
supp, sat, and dep

- Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice $\mathcal{D}(x) = \{A : x(A) = f(A)\}$.

- supp(x) is not nec. tight, but for an extremal point, supp(x) is tight (we see this from definition of extremal point defined by $x(E_i) = f(E_i)$).

- For $x \in P_f$, with x extremal, is supp(x) = sat(x)?

- Consider the case where disjoint $A, B \subseteq E$, we have $f(A) = f(B) = f(A \cup B)$ (meaning perfect dependence / redundancy).

- Suppose $x \in P_f$ has $x(A) > 0$ but $x(B) = 0$.

- Then supp(x) = A

- sat(x) = $\bigcup\{A : x(A) = f(A)\}$ and since $x(A \cup B) = x(A) = f(A) = f(A \cup B)$, here, sat(x) = $A \cup B$.

- In general, sat(x) $\supseteq$ supp(x).

For modular function, we need equality.
Consider $x \in P_f$, and consider the following set
\[ D = \{ \text{dep}(x, e) : e \in \text{sat}(x) \} \] (26)
Consider $x \in P_f$, and consider the following set
\[ \mathcal{D} = \{ \text{dep}(x, e) : e \in \text{sat}(x) \} \] (26)
Moreover, define a partial order on $\mathcal{D}$ as if $A, B \in \mathcal{D}$, then $A \preceq B$ iff $A \subseteq B$. 
Consider $x \in P_f$, and consider the following set
\[
D = \{ \text{dep}(x, e) : e \in \text{sat}(x) \}\]  

(26)

Moreover, define a partial order on $D$ as if $A, B \in D$, then $A \preceq B$ iff $A \subseteq B$.

We’re going to use this partial order to define a partial order on all elements of $\text{sat}(x)$. 
Let $x \in P_f$ again be an extreme point
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.
Let $x \in P_f$ again be an extreme point.
Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).
Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.
Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.
Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$. 
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.

Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$.
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.

Thus, for every pair $a, e \in \text{sat}(x)$, we have that $	ext{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$.

We have just proven
Let $x \in P_f$ again be an extreme point

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.

Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$.

We have just proven

**Theorem 7.1**

*If $x \in P_f$ is an extreme point, then $\preceq$ is a partial order on $\text{sat}(x)$ where for $a, e \in \text{sat}(x)$, the order $\preceq$ is defined by: $a \preceq e$ iff $a \in \text{dep}(x, e)$.*
Scratch Paper
Sources for Today’s Lecture