Announcements

- Homework 2 is due tonight at 11:45pm. All things in lectures marked “exercise”
- Again, all submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.
- Last lecture, all annotations apparently lost (unless you are a PDF expert). Please email me any typos you discover in lecture 14!!
We need to find one makeup lecture this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20): lattices, → SVM
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/9): 3-7:30pm (EEB-303)?
Theorem 2.1

For a given ordering $E = (e_1, \ldots, e_m)$ of $E$ and a given $E_i$ and $x$ generated by $E_i$ using the greedy procedure, then $x$ is an extreme point of $P_f$.

Theorem 2.2

If $x$ is an extreme point of $P_f$ and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$, then $x$ is generated using greedy by some ordering of $B$. 

Partially ordered set

- A partially ordered set (poset) is a set of objects with an order.
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In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):  

For all $x, x \leq x$.  

If $x \leq y$ and $y \leq x$, then $x = y$.  

If $x \leq y$ and $y \leq z$, then $x \leq z$.  

(Reflexive) (P1.)  

(Antisymmetry) (P2.)  

(Transitivity) (P3.)
Partially ordered set

- A partially ordered set (poset) is a set of objects with an order.
- In a poset, for any \( x, y, z \in V \) the following conditions hold (by definition):
  
  For all \( x, x \preceq x \). (Reflexive) \((P1.)\)
  
  If \( x \preceq y \) and \( y \preceq x \), then \( x = y \). (Antisymmetry) \((P2.)\)
  
  If \( x \preceq y \) and \( y \preceq z \), then \( x \preceq z \). (Transitivity) \((P3.)\)

- The order \( n(P) \) of a poset \( P \) is meant the (cardinal) number of its elements.
Partially ordered set

Hasse-diagram: We can draw a poset using a graph where each \( x \in V \) is a node, and if \( x \sqsubseteq y \) we draw \( y \) directly above \( x \) with a connecting edge, but no other edges.

Theorem 2.3
Every non-empty finite subset \( X \subseteq V \) has a minimal (and maximal) element.
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**Theorem 2.3**

*Every non-empty finite subset \( X \subseteq V \) has a minimal (and maximal) element.*
Theorem 2.4

Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.

Proof.

Let $X = \{x_1, \ldots, x_n\}$. Define $m_1 = x_1$ and

$$m_k = \begin{cases} x_k & \text{if } x_k \prec m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases}$$

Then we have constructed $m_n \preceq m_{n-1} \preceq \cdots \preceq m_1$ meaning there is no $m_k$ for $k < n$ such that $m_k \prec m_n$. Let $M = \{m_1, \ldots, m_n\}$. By construction, we also have that there is no $x \in X$ with $x \prec m_n$, thus $m_n$ is minimal.
Given a poset $V$, the length $\ell(V)$ is defined to be the l.u.b. of the lengths of any chains in $V$. That is, $\ell(V)$ is the least upper bound, i.e., smallest number not less than any chain length in $V$. 
Partially ordered set

- Given a poset \( V \), the length \( \ell(V) \) is defined to be the l.u.b. of the lengths of any chains in \( V \). That is, \( \ell(V) \) is the least upper bound, i.e., smallest number not less than any chain length in \( V \).

- The **height** or **dimension** of an element \( x \in V \), or \( l = h(x) \) is the l.u.b. of the lengths of the chains \( 0 = x_0 \prec x_1 \prec \ldots x_l = x \) between 0 and \( x \). Note that \( h(1) = \ell(V) \) when they exist. \( h(x) = 1 \) iff \( 0 \sqsubseteq x \) and such elements (with unit height) are called “atoms” or “points” or “(ground) elements”.
Definition 2.5 (Jordan-Dedekind Chain Condition)

(or JDCC) All maximal length chains between the same endpoints have the same finite length.
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Theorem 2.6
Let $V$ be a poset with $0 \in V$ and where all chains are finite. Then $V$ satisfies JDCC iff it is graded by $h(x)$ (the height function).
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Theorem 2.6
Let $V$ be a poset with $0 \in V$ and where all chains are finite. Then $V$ satisfies JDCC iff it is graded by $h(x)$ (the height function).

- With JDCC, if $x \sqsubseteq y$ then $h(x) + 1 = h(y)$. 
Partially ordered set

- With JDCC, element $x$ has height or rank $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between $x$ and $y$. 
Partially ordered set

- With JDCC, element $x$ has height or rank $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between $x$ and $y$.

- We say a poset is “graded” if it is graded by the height function.
Lattice defined

- Given $X \subseteq V$, $y \in V$ is an upper bound of $X$ if $x \preceq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just $\text{sup}X$), then $y \preceq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \preceq y_2$ and $y_2 \preceq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).
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Definition 2.7 (lattice)

A lattice is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or meet denoted by $x \land y \in V$, and also have a l.u.b. or join denoted by $x \lor y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).
Lattice defined

- Given $X \subseteq V$, $y \in V$ is an upper bound of $X$ if $x \preceq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just sup$X$), then $y \preceq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \preceq y_2$ and $y_2 \preceq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).

**Definition 2.7 (lattice)**

A lattice is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or meet denoted by $x \wedge y \in V$, and also have a l.u.b. or join denoted by $x \vee y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).

- Note again, that such l.u.b.’s and g.l.b.’s are unique if they exist.
Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a \( \pm \infty \) is complete). \( 2^E \) for some set \( E \) is complete. Note that \( E \) can be countably or uncountably infinite.
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- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \lor f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$. 

![Diagram](image-url)
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Any non-empty lattice contains a greatest element \(1 \in V\) and a
least element \(0 \in V\).
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- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.
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- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.

- In a chain, $x \land y$ is the smaller of the two, and $x \lor y$ is the larger of the two.
Definition 2.8 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.
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- Given any $x \preceq y$, then all elements $\{ z : x \preceq z \preceq y \}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.
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A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$). A sublattice is a lattice.

- Given any $x \preceq y$, then all elements $\{z : x \preceq z \preceq y\}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.

- Obviously, $2^E$ for some set $E$ is a lattice, with join/meet being union/intersection. See Figure(C).
Lattices

Theorem 2.9

In any poset \( V \), the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

\[
\begin{align*}
x \land x &= x, \quad x \lor x = x & \text{(Idempotent)} \quad \text{(L1)} \\
x \land y &= y \land x, \quad x \lor y = y \lor x & \text{(Commutative)} \quad \text{(L2)} \\
x \land (y \land z) &= (x \land y) \land z, \quad x \lor (y \lor z) = (x \lor y) \lor z & \text{(Associative)} \quad \text{(L3)} \\
x \land (x \lor y) &= x \lor (x \land y) = x & \text{(Absorption)} \quad \text{(L4)} \\
x \preceq y & \iff x \land y = x \text{ and } x \lor y = y & \text{(Consistency)} \quad \text{(CON)}
\end{align*}
\]

Note the above works for posets, not necessary for it to be a lattice.
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\end{align*}

Note the above works for posets, not necessary for it to be a lattice.
Theorem 2.10

Given a poset $V$ with $0 \in V$, then for all $x \in V$,

$$0 \wedge x = 0 \text{ and } 0 \vee x = x$$

(2)
Lattices

**Theorem 2.10**

*Given a poset* \( V \) *with* \( 0 \in V \), *then for all* \( x \in V \),

\[
0 \land x = 0 \text{ and } 0 \lor x = x
\]  

(2)

**Theorem 2.11**

*In any* lattice, the operations of join and meet are order-preserving in the following sense:

\[
y \leq z \Rightarrow x \land y \leq x \land z \text{ and } x \lor y \leq x \lor z
\]  

(3)
Lattices

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0 \land x = 0 \quad \text{and} \quad 0 \lor x = x
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y \preceq z \Rightarrow x \land y \preceq x \land z \quad \text{and} \quad x \lor y \preceq x \lor z
\]  

(3)

Theorem 2.12

In any lattice, the following distributive inequalities hold for all \( x, y, z \in V \):

\[
x \land (y \lor z) \succeq (x \land y) \lor (x \land z) \tag{4a}
\]

\[
x \lor (y \land z) \succeq (x \lor y) \land (x \lor z) \tag{4b}
\]
Distributive Inequalities

**Theorem 2.13**

In any lattice, the following *distributive inequalities* hold for all \( x, y, z \in V \):

\[
x \land (y \lor z) \geq (x \land y) \lor (x \land z) \tag{5a}
\]

\[
x \lor (y \land z) \leq (x \lor y) \land (x \lor z) \tag{5b}
\]
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

Example:

\[(D)\]

\[
\begin{array}{ccc}
0 & \lor & a \\
& & b \\
& & c \\
& \lor & \\
1 & & \\
\end{array}
\]

\[
(a \land (b \lor c)) = a \land 1 = a
\]

\[
((a \land b) \lor (a \land c)) = 0 \lor 0 = 0
\]

\[
a \prec 0.
\]

Also, in (D) we have

\[
(a \lor (b \land c)) = a \lor 0 = a,
\]

\[
((a \lor b) \land (a \lor c)) = 1 \land 1 = 1.
\]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

For example, in (D), we have that $a \land (b \lor c) = a \land 1 = a$ but $(a \land b) \lor (a \land c) = 0 \lor 0 = 0$ and obviously $a \succ 0$.  

\[ 
\begin{array}{c}
a \\
\downarrow \\
\bullet \\
\downarrow \\
b \\
\downarrow \\
0 \\
\uparrow \\
c \\
\uparrow \\
1 \\
\end{array} 
\]
Distributive Inequalities

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For example, in (D), we have that $a \land (b \lor c) = a \land 1 = a$ but $(a \land b) \lor (a \land c) = 0 \lor 0 = 0$ and obviously $a \succ 0$.

Also, in (D) we have

$a \lor (b \land c) = a \lor 0 = a \prec$

$(a \lor b) \land (a \lor c) = 1 \land 1 = 1$. 
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

\[(E) \quad 0 \quad 1 \quad a \quad b \quad c\]

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Also, in (E), we have

\[b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land (b \lor c) = 1 \land c = c\]
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

In (E), we have that
\[ c \land (a \lor b) = c \land 1 = c \succ (c \land a) \lor (c \land b) = 0 \lor b = b \]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

In (E), we have that
\[ c \land (a \lor b) = c \land 1 = c \triangleright \]
\[ (c \land a) \lor (c \land b) = 0 \lor b = b \]

Also, in (E), we have
\[ b \lor (a \land c)b \lor 0 = b \triangleleft \]
\[ (b \lor a) \land (b \lor c) = 1 \land c = c \]
Modular inequality

**Theorem 2.14**

*In any lattice, the following modular inequalities holds for all $x, y, z \in V$:

$$x \preceq z \Rightarrow x \lor (y \land z) \preceq (x \lor y) \land z$$

(6)
A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.
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Some lattices are such that the distributive inequality is an equality everywhere, and these are called distributive lattices. Only one quality is necessary since:

\[
\begin{align*}
  x \land (y \lor z) &= (x \land y) \lor (x \land z) \\
  x \lor (y \land z) &= (x \lor y) \land (x \lor z)
\end{align*}
\]
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

- Some lattices are such that the distributive inequality is an equality everywhere, and these are called distributive lattices. Only one quality is necessary since:

**Theorem 3.1**

*In any lattice, the following are equivalent:*

\[
x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \quad (7a)
\]

\[
x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \quad (7b)
\]
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

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\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \quad (7b) \]

It is important to note the \( \forall x, y, z \) since this is not true only for individual elements. Note moreover that this means that the operators \( \lor = + \) and \( \land = \cdot \) do not form a lattice over \( \mathbb{R} \).
Theorem 3.2

In any lattice, the following are equivalent:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \quad (8a) \]

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \quad (8b) \]
Distributive Lattices

Theorem 3.2

In any lattice, the following are equivalent:

\[
x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z (8a)
\]

\[
x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z (8b)
\]

Proof.

Take as given the 2nd equation and show the first. Then

\[
(x \land y) \lor (x \land z) = [(x \land y) \lor x] \land [(x \land y) \lor z] \quad \text{by the 2nd eq} (9)
\]

\[
= x \land [(x \land y) \lor z] \quad x \land y \preceq x (10)
\]

\[
= x \land [(x \lor z) \land (y \lor z)] \quad \text{by the 2nd eq} (11)
\]

\[
= x \land (x \lor z) \land (y \lor z) \quad \text{associative} (12)
\]

\[
= x \land (y \lor z) \quad x \lor z \succeq x (13)
\]
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
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- Thus a lattice is distributive if either of the above equalities hold.
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**Example 3.3**

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that $x$ divides $y$. I.e., $2 \preceq 4$ but $2 \not\preceq 5$. Then this is lattice with $x \lor y = \text{l.c.m.}(x, y)$ and $x \land y = \text{g.c.d.}(x, y)$. It is also distributive. Again consider figure (B).
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
- Thus a lattice is distributive if either of the above equalities hold.

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Theorem 3.4 (identity)

In a distributive lattice, if $z \land x = z \land y$ and $z \lor x = z \lor y$ then $x = y$. 
In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

Definition 4.1 (modular identity)

∀ x, y, z, if x ⪯ z, then x ∨ (y ∧ z) = (x ∨ y) ∧ z.

(L5)

Clearly any distributive lattice satisfies the modular identity since when x ⪯ z we have that x ∨ z = z and from the 2nd of the distributive lattice equalities (i.e., x ∨ (y ∧ z) = (x ∨ y) ∧ (x ∨ z)) we get the modular identity.

Easy way to remember.

x, y, z and x ∨ (y ∧ z) = (x ∨ y) ∧ z.

The term "modular" comes from abstract algebra, where a R-module is an abstract system that generalizes (R, R^n) (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this identity.
In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 4.1 (modular identity)**

\[ \forall x, y, z, \text{ if } x \preceq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \] (L5)

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Easy way to remember. \( x, y, z \) and \( x \lor (y \land z) = (x \lor y) \land z \)

The term “modular” comes from abstract algebra, where a R-module is an abstract system that generalizes \((\mathbb{R}, \mathbb{R}^n)\) (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this identity.
Not every lattice is modular. Figure (D) is modular but not distributive. We already saw that (D) is not distributive since it is strict for certain assignments. It is modular though.

![Diagram of modular lattice](image-url)
Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land c = 1 \land c = c$, so modular equality is violated.
Modular Lattices

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**Theorem 4.2**

Any non-modular lattice \( V \) contains the lattice in Figure (E) as a sublattice.
Modular Lattices

- Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \lor (a \land c) = b \lor 0 = b \preceq (b \lor a) \land c = 1 \land c = c$, so modular equality is violated.

Theorem 4.2

Any non-modular lattice $V$ contains the lattice in Figure (E) as a sublattice.

- Thus, the structure (E) is fundamental to non-modular lattices.
Modular Lattices

Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

Upper-Semimodularity if $x$ and $y$ cover $z$ and $x \neq y$ then $x \lor y$ covers both $x$ and $y$,

Lower-Semimodularity if $z$ covers $x$ and $y$ and $x \neq y$ then $x$ and $y$ both covers $x \land y$.

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if $x$ and $y$ cover $z$ and $x \neq y$ then $x \lor y$ covers both $x$ and $y$, and

Thus, upper-semimodularity means that if $z \sqsubseteq x$ and $z \sqsubseteq y$, and if $x \neq y$, then $x \sqsubseteq (x \lor y)$ and $y \sqsubseteq (x \lor y)$. 
Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if \( x \) and \( y \) cover \( z \) and \( x \neq y \) then \( x \lor y \) covers both \( x \) and \( y \), and

**Lower-Semimodularity** if \( z \) covers \( x \) and \( y \) and \( x \neq y \) then \( x \) and \( y \) both covers \( x \land y \).

Thus, lower-semimodularity means that if \( x \sqsubseteq z \) and \( y \sqsubseteq z \), and if \( x \neq y \), then \( (x \land y) \sqsubseteq x \) and \( (x \land y) \sqsubseteq y \).
Modular Lattices

Theorem 4.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if \( x \) and \( y \) cover \( z \) and \( x \neq y \) then \( x \lor y \) covers both \( x \) and \( y \), and

**Lower-Semimodularity** if \( z \) covers \( x \) and \( y \) and \( x \neq y \) then \( x \) and \( y \) both cover \( x \land y \).

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
Theorem 5.1

Let $L$ be a finite lattice. The following two conditions are equivalent:

(i) $L$ is graded, and the height function $h(\cdot)$ of $L$ satisfies the (what we know as the submodular) inequality for all $x, y \in L$.

$$h(x) + h(y) \geq h(x \lor y) + h(x \land y)$$ (14)

(ii) If $x$ and $y$ both cover $z$, then $x \lor y$ covers both $x$ and $y$.
Semi-modular/Submodular Lattices: \((i) \implies (ii)\)

\[ h_{\text{submodular}} \implies \left\{ (z \subseteq x, z \subseteq y) \implies (x \subseteq (x \vee y)), (y \subseteq (x \vee y)) \right\} . \]

- Suppose \(x\) and \(y\) cover \(z\).

...
Semi-modular/Submodular Lattices: (i) ⇒ (ii)

\[ h \text{ submodular} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} . \]

- Suppose \( x \) and \( y \) cover \( z \).
- Note that if \( x \) and \( y \) cover \( z \) then since \( L \) is a lattice, \( z = x \land y \).
Semi-modular/Submodular Lattices: \((i) \Rightarrow (ii)\)

\(h\) submodular \(\Rightarrow \left\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\right\} \).

- Suppose \(x\) and \(y\) cover \(z\).
- Note that if \(x\) and \(y\) cover \(z\) then since \(L\) is a lattice, \(z = x \land y\).
- Then we have \(h(x) = h(y) = h(x \land y) + 1\).
Semi-modular/Submodular Lattices: \((i) \Rightarrow (ii)\)

\[ h \text{ submodular} \Rightarrow \left\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\right\}. \]

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- Then we have \(h(x) = h(y) = h(x \land y) + 1\).
- Also, since \(x\) and \(y\) are distinct, and since they both cover \(z\) we can’t have (w.l.o.g.) \(x \leq y\), and thus \(h(x \lor y) > h(x) = h(y)\).

\[ h \text{ submodular} \Rightarrow \left\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\right\}. \]
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- Hence by \((i)\), we have
  \[ h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \] (15)

...
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\[
h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \tag{15}
\]

or
\[
h(x \land y) + 2 \geq h(x \lor y) > h(x \land y) + 1 \tag{16}
\]
Semi-modular/Submodular Lattices: \((i) \Rightarrow (ii)\)

\[ h \text{ submodular} \Rightarrow \left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \]

- Suppose \(x\) and \(y\) cover \(z\).
- Note that if \(x\) and \(y\) cover \(z\) then since \(L\) is a lattice, \(z = x \land y\).
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  h(x \land y) + 2 \geq h(x \lor y) > h(x \land y) + 1 \tag{16}
  \]
  giving \(h(x \lor y) = h(x \land y) + 2 = h(x) + 1 = h(y) + 1\), so that \(x \lor y\) covers both \(x\) and \(y\).
Semi-modular/Submodular Lattices: \((\text{ii}) \Rightarrow (\text{i})\)

\[
\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \} \Rightarrow h \text{ submodular.}
\]

Suppose \(L\) is not graded, and let \([u, v]\) be an interval of \(L\) of minimal length that is not graded (so all smaller length intervals are graded).
Semi-modular/Submodular Lattices: \( (\text{ii}) \Rightarrow (\text{i}) \)

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- Then there are elements \( x_1, x_2 \) of \([u, v]\) where each of \( x_1 \) and \( x_2 \) cover \( u \),
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

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- Suppose \(L\) is not graded, and let \([u, v]\) be an interval of \(L\) of minimal length that is not graded (so all smaller length intervals are graded).
- Then there are elements \(x_1, x_2\) of \([u, v]\) where each of \(x_1\) and \(x_2\) cover \(u\),
- By the minimality, all maximal chains of each interval \([x_i, v]\) have the same length \(l_i\) where \(l_1 \neq l_2\).
Semi-modular/Submodular Lattices: \( (ii) \Rightarrow (i) \)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \vee y)), (y \sqsubseteq (x \vee y)) \right\} \Rightarrow h \text{ submodular.}
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- By the minimality, all maximal chains of each interval \([x_i, v]\) have the same length \( \ell_i \) where \( \ell_1 \neq \ell_2 \).
- By \((ii)\), there are saturated chains in \([x_i, v]\) of the form \( x_i \prec x_1 \vee x_2 \prec y_1 \prec y_2 \prec \cdots \prec y_k = v \), contradicting \( \ell_1 \neq \ell_2 \).
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

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\left\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\right\} \Rightarrow h \text{ submodular.}
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- By the minimality, all maximal chains of each interval \([x_i, v]\) have the same length \(\ell_i\) where \(\ell_1 \neq \ell_2\).
- By \((ii)\), there are saturated chains in \([x_i, v]\) of the form \(x_i \prec x_1 \lor x_2 \prec y_1 \prec y_2 \prec \cdots \prec y_k = v\), contradicting \(\ell_1 \neq \ell_2\).
- Hence \(L\) is graded (i.e., every maximal chain has the same length, i.e., JDCC holds).
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Now suppose there is a pair \(x, y \in L\) violating the submodularity inequality, i.e., with
  \[
  h(x) + h(y) < h(x \lor y) + h(x \land y)
  \] (17)
  and choose such a pair first with \(\ell(x \land y, x \lor y)\) minimal, and then (second) with \(h(x) + h(y)\) minimal.
Semi-modular/Submodular Lattices: (ii) \(\Rightarrow\) (i)

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- By (ii), we cannot have both \(x\) and \(y\) covering \(x \land y\) (because if we did, then \(h(x) = h(x \land y) + 1\), \(h(y) = h(x \land y) + 1\), and (ii) gives that \(h(x \lor y) = h(x) + 1 = h(y) + 1\), and we would have the submodular inequality at equality).
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\begin{align*}
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} & \Rightarrow h \text{ submodular.}
\end{align*}
\]

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- Thus assume that \(x \land y \prec x' \prec x\), say (w.l.o.g.)
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\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
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  and choose such a pair first with $\ell(x \land y, x \lor y)$ minimal, and then (second) with $h(x) + h(y)$ minimal.

- By (ii), we cannot have both $x$ and $y$ covering $x \land y$ (because if we did, then $h(x) = h(x \land y) + 1$, $h(y) = h(x \land y) + 1$, and (ii) gives that $h(x \lor y) = h(x) + 1 = h(y) + 1$, and we would have the submodular inequality at equality).

- Thus assume that $x \land y \prec x' \prec x$, say (w.l.o.g.)

- By the minimality of $\ell(x \land y, x \lor y)$ and $h(x) + h(y)$, we have
  \[
h(x') + h(y) \geq h(x' \land y) + h(x' \lor y).
  \]
Semi-modular/Submodular Lattices: \( (ii) \Rightarrow (i) \)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Now \( x' \land y = x \land y \), so Eq. 17 and Eq. 18 together imply that
  \[
h(x) + h(x' \lor y) < h(x') + h(x \lor y). \tag{19}
\]
Semi-modular/Submodular Lattices: \((ii) \implies (i)\)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \implies (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \implies h \text{ submodular.}
\]

- Now \(x' \land y = x \land y\), so Eq. 17 and Eq. 18 together imply that
  \[
  h(x) + h(x' \lor y) < h(x') + h(x \lor y).
  \] (19)
- Since \(x \succ x'\), we have \(x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y\).
Semi-modular/Submodular Lattices: (ii) \(\Rightarrow\) (i)

\[
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  \[
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  \]  
  (19)

- Since \(x \succ x'\), we have \(x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y\).

- Also, by the modular inequalities (with \(x \leftarrow x', y \leftarrow y, z \leftarrow x\)), we have \(x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x'\).
\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \} \Rightarrow h \text{ submodular.}

- Now $x' \land y = x \land y$, so Eq. 17 and Eq. 18 together imply that
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  \hspace{1cm} (19)

- Since $x \succ x'$, we have $x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y$.

- Also, by the modular inequalities (with $x \leftrightarrow x'$, $y \leftrightarrow y$, $z \leftrightarrow x$), we have $x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x'$.

- Hence setting $X = x$, $Y = x' \lor y$. This gives $X \lor Y = x \lor y$, and $X \land Y \succeq x' \succ x$. 

...
Semi-modular/Submodular Lattices: \((ii) \Rightarrow (i)\)

\[
\left\{ (z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y)) \right\} \Rightarrow h \text{ submodular.}
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- Hence setting \(X = x, Y = x' \lor y\). This gives \(X \lor Y = x \lor y\), and \(X \land Y \succeq x' \succ x\).

- Thus, we have found a pair \(X, Y \in L\) with
  \[
  h(X) + h(Y) < h(X \land Y) + h(X \lor Y)
  \]
  and a strictly shorter length
  \[
  \ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y),
  \]
Semi-modular/Submodular Lattices: (ii) ⇒ (i)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
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  \[ (19) \]
- Since \( x \succ x' \), we have \( x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y \).
- Also, by the modular inequalities (with \( x \leftrightarrow x', y \leftrightarrow y, z \leftrightarrow x \)), we have \( x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x' \).
- Hence setting \( X = x, Y = x' \lor y \). This gives \( X \lor Y = x \lor y \), and \( X \land Y \succeq x' \succ x \).
- Thus, we have found a pair \( X, Y \in L \) with
  \[ h(X) + h(Y) < h(X \land Y) + h(X \lor Y) \]
  and a strictly shorter length
  \[ \ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y), \]
- This contradicts the minimality of \( \ell(x \land y, x \lor y) \).

...
Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

\[
\{(z \sqsubseteq x, z \sqsubseteq y) \Rightarrow (x \sqsubseteq (x \lor y)), (y \sqsubseteq (x \lor y))\} \Rightarrow h \text{ submodular.}
\]

- Now $x' \land y = x \land y$, so Eq. 17 and Eq. 18 together imply that
\[
h(x) + h(x' \lor y) < h(x') + h(x \lor y).
\]
(19)

- Since $x \succ x'$, we have $x \lor (x' \lor y) = (x \lor x') \lor y = x \lor y$.

- Also, by the modular inequalities (with $x \leftrightarrow x'$, $y \leftrightarrow y$, $z \leftrightarrow x$), we have $x \land (x' \lor y) \succeq x' \lor (y \land x) \succeq x'$.

- Hence setting $X = x$, $Y = x' \lor y$. This gives $X \lor Y = x \lor y$, and $X \land Y \succeq x' \succ x$.

- Thus, we have found a pair $X, Y \in L$ with
\[
h(X) + h(Y) < h(X \land Y) + h(X \lor Y)
\]
and a strictly shorter length
\[
\ell(X \land Y, X \lor Y) < \ell(x \land y, x \lor y),
\]

- This contradicts the minimality of $\ell(x \land y, x \lor y)$.

- The proof is complete.
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

This lattice is not modular since $x \lor y$ covers $x$ and $y$, but $x$ and $y$ don’t cover $x \land y$.

- $2 + 2 > 3 + 0$
- Submodularity:
  - $h(x) + h(y) > h(x \lor y) + h(x \land y)$
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

Also note, this violates the modular equality

\[ (\forall x, y, z, \; x \preceq z \Rightarrow (x \lor (y \land z) = (x \lor y) \land z) \). \]
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

Also note, this violates the modular equality

$$(\forall x, y, z, \quad x \preceq z \Rightarrow (x \lor (y \land z) = (x \lor y) \land z))$$

Flip it up side down to get a lower-semimodular (or “supermodular”) lattice.
Definition 6.1 (ideal)

An ideal is a nonvoid subset $J$ of a lattice $L$ with the properties

$$\forall a \in J, x \in L, \ x \preceq a \Rightarrow x \in J$$  \hspace{1cm} (20)

$$\forall a \in J, \ b \in J \Rightarrow a \lor b \in J.$$  \hspace{1cm} (21)

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).
Ideal in a Lattice

**Definition 6.1 (ideal)**

An ideal is a nonvoid subset $J$ of a lattice $L$ with the properties

$$\forall a \in J, x \in L, \ x \leq a \Rightarrow x \in J$$  \hspace{1cm} (20)

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The dual concept (in a lattice) is called a dual ideal (or a meet ideal).

**Proposition 6.2**

$J$ is an ideal when $a \lor b \in J$ iff $a \in J, b \in J$ (closure under join).
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Proposition 6.2

$J$ is an ideal when $a \lor b \in J$ iff $a \in J, \ b \in J$ (closure under join).

Example 6.3

In $2^E$, take any $A \subseteq E$, then $L(A) = \{B : B \subseteq A\}$ is an ideal in a set lattice.
Definition 6.4

Given an element $a \in L$ in a lattice, the set $L(a)$ of all elements $\{x : x \preceq a, x \in L\}$ is an ideal, and is called a principle ideal.
Ideal in a Lattice

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In fact, in any finite lattice, every (nonvoid) ideal is a principle ideal. In fact, we have:
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In fact, in any finite lattice, every (nonvoid) ideal is a principle ideal. In fact, we have:

Theorem 6.5

*The set of all ideals’ of any lattice \( L \), ordered by inclusion, itself forms a lattice. The set of all principal ideals in \( L \) forms a sublattice of this lattice, which is isomorphic with \( L \).*
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The set of all ideals of any lattice \( L \), ordered by inclusion, itself forms a lattice. The set of all principal ideals in \( L \) forms a sublattice of this lattice, which is isomorphic with \( L \).

Example 6.6

Consider \( 2^E \). Then for any \( A \subseteq E \), we see that \( L(A) = \{B : B \subseteq A\} \) is an ideal. Also, we can see that the set of sets \( \{L(A) : A \subseteq E\} \) is isomorphic to \( 2^E \) and also forms a lattice.
Definition 6.7

A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \lor y = 1$ and $x \land y = 0$. 
Complement and Complemented Lattices

Definition 6.7

A lattice with a 0 and 1 is **complemented** if for all \( x \in L \) there exists a \( y \in L \) such that \( x \lor y = 1 \) and \( x \land y = 0 \). A lattice is **relatively complemented** if every interval \([x, y]\) is complemented (w.r.t. the interval, with \( x \) taking the role of 0 and \( y \) taking the role of 1).
Complement and Complemented Lattices

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A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \lor y = 1$ and $x \land y = 0$. A lattice is relatively complemented if every interval $[x, y]$ is complemented (w.r.t. the interval, with $x$ taking the role of 0 and $y$ taking the role of 1).

Recall, an atom of a finite lattice is an element covering 0.
Definition 6.7

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Any complemented modular lattice is relatively complemented.
Complement and Complemented Lattices

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A lattice with a 0 and 1 is **complemented** if for all \( x \in L \) there exists a \( y \in L \) such that \( x \lor y = 1 \) and \( x \land y = 0 \). A lattice is **relatively complemented** if every interval \([x, y]\) is complemented (w.r.t. the interval, with \( x \) taking the role of 0 and \( y \) taking the role of 1).

Recall, an atom of a finite lattice is an element covering 0.

**Proposition 6.8**

Any complemented modular lattice is relatively complemented.

**Proposition 6.9**

*In a complemented modular lattice of finite length, every element is the join of those elements which it contains.*
Definition 6.10

A **boolean lattice** is a complemented distributive lattice.
### Boolean Lattices

**Definition 6.10**

A **boolean lattice** is a complemented distributive lattice.

**Theorem 6.11**

In any boolean lattice, each element $x$ has a unique complement $x'$. Moreover, we have

\[
\begin{align*}
x \land x' &= 0, \quad x \lor x' = 1 \\
(x')' &= x, \\
(x \land y)' &= x' \lor y', \quad (x \lor y)' = x' \land y'
\end{align*}
\]  

(L1)  
(L2)  
(L3)
Join Irreducible

**Definition 6.12**

An element $x$ of a lattice is called **join irreducible** if $y \lor z = x$ implies $y = x$ or $z = x$ (i.e., if $x$ is the join of two elements, it must be one of those elements).
Join Irreducible

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Proposition 6.13

*If all chains in a lattice are finite, then every $a \in L$ can be represented as a join $a = x_1 \lor \ldots \lor x_n$ of a finite number of join irreducible elements.*
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Proposition 6.13

*If all chains in a lattice are finite, then every $a \in L$ can be represented as a join $a = x_1 \lor \ldots \lor x_n$ of a finite number of join irreducible elements.*

Proposition 6.14

*In any complemented modular lattice, all join irreducible elements are atoms.*
Definition 6.15 (ring family)

A **ring of sets** (or **ring family**) is a family $\Phi$ of subsets of a set $E$ which contains with any two sets $S$ and $T$ also their (set-theoretic) intersection $S \cap T$ and union $S \cup T$. A **field of sets** is a ring of sets which contains with any $S$ also its set complement $E \setminus S$.
Definition 6.15 (ring family)

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Thus, any **ring** of sets under the natural ordering $S \subset T$ forms a distributive lattice.
Theorem 6.16

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L \simeq 2^X$. 

Join irreducible, ground elements, Boolean lattices
Join irreducible, ground elements, Boolean lattices

**Theorem 6.16**

Let $L$ be any distributive lattice of length $n$. Then the poset $X$ of join-irreducible elements $x \succ 0$ has order $n$ and, moreover, $L \cong 2^X$.

- The join-irreducible elements of a distributive lattice constitute a form of set of “ground elements” which generate the distributive lattice.
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Theorem 6.16

Let \( L \) be any distributive lattice of length \( n \). Then the poset \( X \) of join-irreducible elements \( x \succ 0 \) has order \( n \) and, moreover, \( L \cong 2^X \)

- The join-irreducible elements of a distributive lattice constitute a form of set of “ground elements” which generate the distributive lattice.
- Thus, any distributive lattice of length \( n \) is isomorphic with a ring of subsets of a set \( E \) of \( n \) elements.
- The next result is perhaps not so surprising.
The join-irreducible elements of a distributive lattice constitute a form of set of “ground elements” which generate the distributive lattice.

Thus, any distributive lattice of length \( n \) is isomorphic with a ring of subsets of a set \( E \) of \( n \) elements.

The next result is perhaps not so surprising.

Theorem 6.17

Every Boolean lattice of finite length \( n \) is isomorphic with the field of all subsets of a set of \( |E| = n \) elements, namely \( 2^E \).
supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{e : x(e) \neq 0\}$
supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{ e : x(e) \neq 0 \}$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e.,
  \[ \text{sat}(x) = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}. \]
  That is,
  \[
  \text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{ A : A \in D(x) \} \]
  \[= \bigcup \{ A : A \subseteq E, x(A) = f(A) \} \] (22)

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\[
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\text{cl}(x) & \overset{\text{def}}{=} \text{sat}(x) = \bigcup \{ A : A \in \mathcal{D}(x) \} \\
& = \bigcup \{ A : A \subseteq E, x(A) = f(A) \} \quad (23) \\
& = \{ e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f \} \\
\end{align*}
\]

- For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated ($x$-tight) set w.r.t. $x$ containing $e$. That is,

\[
\text{dep}(x, e) = \begin{cases} 
\bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\
\emptyset & \text{else}
\end{cases}
\]

Prof. Jeff Bilmes
Now, sat(\(x\)) is tight, and corresponds to the largest member of the distributive lattice \(D(x) = \{A : x(A) = f(A)\}\).
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supp\( (x) \) is not nec. tight, but for an extremal point, supp\( (x) \) is tight (we see this from definition of extremal point defined by \( x(E_i) = f(E_i) \)).
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For \( x \in P_f \), with \( x \) extremal, is \( \text{supp}(x) = \text{sat}(x) \)?
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For $x \in P_f$, with $x$ extremal, is supp(x) = sat(x)?

Consider an example case where disjoint $A, B \subseteq E$, we have $f(A) = f(B) = f(A \cup B)$ (meaning perfect dependence / redundancy).
Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( \mathcal{D}(x) = \{A : x(A) = f(A)\} \).

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Suppose \( x \in P_f \) has \( x(A) > 0 \) but \( x(V \setminus A) = 0 \) and so \( x(B) = 0 \).
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- Now, sat($x$) is tight, and corresponds to the largest member of the distributive lattice $\mathcal{D}(x) = \{A : x(A) = f(A)\}$.
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Now, sat(x) is tight, and corresponds to the largest member of the distributive lattice \( D(x) = \{ A : x(A) = f(A) \} \).

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sat(x) = \( \cup \{ A : x(A) = f(A) \} \) and since \( x(A \cup B) = x(A) = f(A) = f(A \cup B) \), here, sat(x) = \( A \cup B \) (at least).
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In general, for extremal \( x \), \( \text{sat}(x) \supseteq \text{supp}(x) \).
Now, sat(\(x\)) is tight, and corresponds to the largest member of the distributive lattice \(D(x) = \{A : x(A) = f(A)\}\).

supp(\(x\)) is not nec. tight, but for an extremal point, supp(\(x\)) is tight (we see this from definition of extremal point defined by \(x(E_i) = f(E_i)\)).

For \(x \in P_f\), with \(x\) extremal, is supp(\(x\)) = sat(\(x\))?  
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sat(\(x\)) = \(\cup\{A : x(A) = f(A)\}\) and since \(x(A \cup B) = x(A) = f(A) = f(A \cup B)\), here, sat(\(x\)) = \(A \cup B\) (at least).

In general, for extremal \(x\), sat(\(x\)) \(\supseteq\) supp(\(x\)).

For modular functions, they are always equal.
Consider $x \in P_f$, and consider the following set

$$D = \{ \text{dep}(x, e) : e \in \text{sat}(x) \}$$

(26)
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Moreover, define a partial order on $D$ as if $A, B \in D$, then $A \preceq B$ iff $A \subseteq B$. 
Consider $x \in P_f$, and consider the following set

$$D = \{ \text{dep}(x, e) : e \in \text{sat}(x) \}$$  \hspace{1cm} (26)

Moreover, define a partial order on $D$ as if $A, B \in D$, then $A \preceq B$ iff $A \subseteq B$.

We’re going to use this partial order to define a partial order on all elements of $\text{sat}(x)$. 
Let $x \in P_f$ again be an extreme point.
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Recall, the equation for \( x \) is of the form \( x(e) = 0 \) for some \( e \) and \( x(A) = f(A) \) for some \( A \) (see earlier).
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.
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Then, for each pair \( a, e \in \text{supp}(x) \), there is a tight set containing w.l.o.g. one of \( a \) or \( e \) but not the other.

Also, for polymatroidal \( f \), we have that for each \( e \in \text{sat}(x) \setminus \text{supp}(x) \), the set \( \text{supp}(x) + e \) is also tight.
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Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$. 
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Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$. 
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Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.

Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$.

We have just proven
Let $x \in P_f$ again be an extreme point.

Recall, the equation for $x$ is of the form $x(e) = 0$ for some $e$ and $x(A) = f(A)$ for some $A$ (see earlier).

Then, for each pair $a, e \in \text{supp}(x)$, there is a tight set containing w.l.o.g. one of $a$ or $e$ but not the other.

Also, for polymatroidal $f$, we have that for each $e \in \text{sat}(x) \setminus \text{supp}(x)$, the set $\text{supp}(x) + e$ is also tight.

Thus, for every pair $a, e \in \text{sat}(x)$, we have that $\text{dep}(x, a) \neq \text{dep}(x, e)$.

And thus the partial order on $\text{dep}(x, e)$ can be used to define a partial order on $\text{sat}(x)$.

We have just proven

**Theorem 7.1**

If $x \in P_f$ is an extreme point, then $\preceq$ is a partial order on $\text{sat}(x)$ where for $a, e \in \text{sat}(x)$, the order $\preceq$ is defined by: $a \preceq e$ iff $a \in \text{dep}(x, e)$. 
Sources for Today’s Lecture