EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
Spring Quarter, 2011
http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 13 - May 13th, 2011
Announcements

- On Final projects. **One** single page final project updates due next Wednesday, 5/18 at 11:45pm.
- Again, all submissions must be done electronically, via our drop box. See the link https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.
- Homework 2 is due next Friday night at 11:45pm. All things in lectures marked “exercise”
We need to find one makeup lectures this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L13 (5/13): More polymatroids, start lattices
- L14 (5/18): lattices/submodular
- L15 (5/20):
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/?): (need to find time/date/place).
Theorem 2.1

For a given ordering $E = (e_1, \ldots, e_m)$ of $E$ and a given $E_i$ and $x$ generated by $E_i$ using the greedy procedure, then $x$ is an extreme point of $P_f$. 
**Polymatroid extreme points**

**Theorem 2.1**

For a given ordering \( E = (e_1, \ldots, e_m) \) of \( E \) and a given \( E_i \) and \( x \) generated by \( E_i \) using the greedy procedure, then \( x \) is an extreme point of \( P_f \).
Polymatroid extreme points

**Theorem 2.1**

For a given ordering $E = (e_1, \ldots, e_m)$ of $E$ and a given $E_i$ and $x$ generated by $E_i$ using the greedy procedure, then $x$ is an extreme point of $P_f$.

Moreover, we have

**Corollary 2.2**

If $x$ is an extreme point of $P_f$ and $B \subseteq E$ is given such that $\{e \in E : x(e) \neq 0\} \subseteq B \subseteq \bigcup (A : x(A) = f(A))$, then $x$ is generated using greedy by some ordering of $B$. 
Recall closure from Lecture 3: Given $A \subseteq E$, the closure or span of $A$, is defined by $\text{span}(A) = \{ b \in E : r(A \cup \{b\}) = r(A) \}$ where $r$ is matroid rank.
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Again, tight sets are closed under union and intersection (Lecture 7), and therefore form a distributive lattice.
Polymatroid Closure/Sat

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- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$, which can constitute a join and meet.
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- For a given $x \in P_f$, we can define this family

  \[
  \mathcal{D}(x) = \{ A : A \subseteq E, x(A) = f(A) \} \tag{1}
  \]
Now given:

\[ \mathcal{D}(x) = \{ A : A \subseteq E, x(A) = f(A) \} \]  
\[ = \{ A : f(A) - x(A) = 0 \} \]
Polymatroid Closure/Sat

- Now given:

\[ \mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \quad (2) \]
\[ = \{A : f(A) - x(A) = 0\} \quad (3) \]

- Since \( x \in P_f \) and \( f \) is presumed to be polymatroid function, we see

\( f'(A) = f(A) - x(A) \) is a non-negative submodular function, and

\( \mathcal{D}(x) \) are the zero-valued minimizers (if any) if \( f'(A) \).

---

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Polymatroid Closure/Sat

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- The zero-valued minimizers of \( f' \) are thus closed under union and intersection. In fact, for arbitrary submodular \( f \), the minimizers are so closed. Let \( A, B \in \text{argmin}_{X \subseteq E} f(X) \).
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The zero-valued minimizers of \( f' \) are thus closed under union and intersection. In fact, for arbitrary submodular \( f \), the minimizers are so closed. Let \( A, B \in \text{argmin}_{X \subseteq E} f(X) \).

Then since \( f(A) = f(B) \leq f(A \cap B) \) and \( f(A) = f(B) \leq f(A \cup B) \), and by submodularity

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

we must have \( f(A) = f(B) = f(A \cup B) = f(A \cap B) \).
Polymatroid Closure/Sat

Matroid closure can be generalized (also called the polymatroid closure or saturation function) as unique maximal element in $\mathcal{D}(x)$. That is, for some $x \in P_f$, we define:

$$
\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\}
$$

(5)

$$
= \bigcup \{A : A \subseteq E, x(A) = f(A)\}
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(6)
Matroid closure can be generalized (also called the polymatroid closure or saturation function) as unique maximal element in $D(x)$. That is, for some $x \in P_f$, we define:

$$\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{A : A \in D(x)\} = \bigcup \{A : A \subseteq E, x(A) = f(A)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\}$$ (5, 6, 7)
Matroid closure can be generalized (also called the polymatroid closure or saturation function) as unique maximal element in $D(x)$. That is, for some $x \in P_f$, we define:

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$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\}$$  \hspace{1cm} (6)$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$$  \hspace{1cm} (7)$$

This generalizes matroid closure in the following way (see next slide):
Consider matroid \((E, \mathcal{I}) = (E, r)\), some \(I \in \mathcal{I}\). Then \(1_I \in P_r\) and 
\[
\mathcal{D}(1_I) = \{A : 1_I(A) = r(A)\},
\]
and
\[
sat(1_I) = \bigcup \{A : A \subseteq E, A \in \mathcal{D}(1_I)\} \quad (8)
\]
\[
= \bigcup \{A : A \subseteq E, 1_I(A) = r(A)\} \quad (9)
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\[
= \bigcup \{A : A \subseteq E, |I \cap A| = r(A)\} \quad (10)
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Consider matroid \((E, \mathcal{I}) = (E, r)\), some \(I \in \mathcal{I}\). Then \(1_I \in P_r\) and 
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Given \(A \in \mathcal{D}(1_I)\), and \(b \in \text{span}(A) \setminus A\), then 
\[ r(A + b) = r(A) = |I \cap A| = r(I \cap A) \] (11)
so \(b \notin I \setminus A\) (since otherwise it would increase the rank) and so 
\[ |I \cap (A + b)| = |I \cap A|, \] meaning \(A + b \in \mathcal{D}(1_I)\).
Consider matroid \((E, \mathcal{I}) = (E, r)\), some \(I \in \mathcal{I}\). Then \(1_I \in P_r\) and \(D(1_I) = \{ A : 1_I(A) = r(A) \}\), and
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sat(1_I) = \bigcup \{ A : A \subseteq E, A \in D(1_I) \} \tag{8}
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Thus, \(sat(1_I) \supseteq \text{span}(I)\).
Polymatroid Closure/Sat

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\mathcal{D}(1_I) = \{A : 1_I(A) = r(A)\} \text{, and} 
\]
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\text{sat}(1_I) = \bigcup \{A : A \subseteq E, A \in \mathcal{D}(1_I)\} 
\]
\[
= \bigcup \{A : A \subseteq E, 1_I(A) = r(A)\} 
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|I \cap (A + b)| = |I \cap A|, \text{ meaning } A + b \in \mathcal{D}(1_I). 
\]

Thus, \(\text{sat}(1_I) \supseteq \text{span}(I)\).

Now, consider \(b \in \text{sat}(1_I) \setminus I\). Choose any \(A \in \mathcal{D}(1_I)\) with \(b \in A\). Then
\[
r(A) = r((I \cap A) \cup (A \setminus I)) = r(I \cap A) = r((I \cap A) + b). 
\]
Polymatroid Closure/Sat

Consider matroid \((E, \mathcal{I}) = (E, r)\), some \(I \in \mathcal{I}\). Then \(1_I \in P_r\) and 
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Now, consider \(b \in sat(1_I) \setminus I\). Choose any \(A \in D(1_I)\) with \(b \in A\).
Then
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r(A) = r((I \cap A) \cup (A \setminus I)) = r(I \cap A) = r((I \cap A) + b).
\]
Thus, \(sat(1_I) \subseteq \text{span}(I)\). Hence \(sat(1_I) = \text{span}(I)\)
Now, consider a matroid \((E, r)\) and some \(B \subseteq E\) with \(B \notin \mathcal{I}\), and consider \(1_B\) (which may not be a vertex of or even a member of \(P_r\)).
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The definition of the following form

\[
\text{sat}(1_B) = \bigcup \{A : A \subseteq E, |A \cap B| = r(A)\}
\]

could be meaningful if we defined it as follows.
Now, consider a matroid \((E, r)\) and some \(B \subseteq E\) with \(B \notin \mathcal{I}\), and consider \(1_B\) (which may not be a vertex of or even a member of \(P_r\)).

The a definition of the following form

\[
\text{sat}(1_B) = \bigcup \{ A : A \subseteq E, \ |A \cap B| = r(A) \} \tag{12}
\]

could be meaningful if we defined it as follows.

Let \(I_B \in \mathcal{I}\) be such that \(I_B \subseteq B \subseteq \text{span}(I_B)\), so \(r(B) = r(I_B)\). Then, make the definition:

\[
\text{sat}(1_B) = \text{sat}(1_{I_B}) \tag{13}
\]

In which case, we also get \(\text{sat}(1_B) = \text{span}(B)\).
Support of vector

- The support of a vector $x \in P_f$ is defined as the elements with non-zero entries.
Support of vector

- The **support** of a vector $x \in P_f$ is defined as the elements with non-zero entries.
- That is

$$\text{supp}(x) = \{e \in E : x(e) \neq 0\}$$  \hspace{1cm} (14)
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The support of a vector $x \in P_f$ is defined as the elements with non-zero entries.

That is

$$\text{supp}(x) = \{ e \in E : x(e) \neq 0 \} \quad (14)$$

Clearly, sat($x$) is a tight set, but not necessarily supp($x$).

For any extremal point $x$, we have supp($x$) is tight since extremal points are defined as a system of equalities of the form $x(E_i) = f(E_i)$ as we saw earlier in lecture.
Dependence Function

- Tight sets can be restricted to contain a particular element. Given $x \in P_f$, and $e \in \text{sat}(x)$, define
  \[
  D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \}
  \]
  \[
  = D(x) \cap \{ A : A \subseteq E, e \in A \}
  \]
  (15) (16)
Dependence Function

- Tight sets can be restricted to contain a particular element. Given \( x \in P_f \), and \( e \in \text{sat}(x) \), define
  \[
  \mathcal{D}(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \tag{15}
  \]
  \[
  = \mathcal{D}(x) \cap \{ A : A \subseteq E, e \in A \} \tag{16}
  \]
- Thus, \( \mathcal{D}(x, e) \subseteq \mathcal{D}(x) \), and \( \mathcal{D}(x, e) \) is a sublattice of \( \mathcal{D}(x) \).
Dependence Function

- Tight sets can be restricted to contain a particular element. Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \}$$

$$= D(x) \cap \{ A : A \subseteq E, e \in A \}$$

(15)

(16)

- Thus, $D(x, e) \subseteq D(x)$, and $D(x, e)$ is a sublattice of $D(x)$.

- Therefore, we can define a unique minimal element of $D(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

(17)
Dependence Function

- Tight sets can be restricted to contain a particular element. Given $x \in P_f$, and $e \in \text{sat}(x)$, define
  \[ D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \]
  \[ = D(x) \cap \{ A : A \subseteq E, e \in A \} \]

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- Therefore, we can define a unique minimal element of $D(x, e)$ denoted as follows:
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  \emptyset & \text{else}
  \end{cases} \]

- I.e., $\text{dep}(x, e)$ is the minimal element in $D(x)$ that contains $e$ (the minimal $x$-tight set containing $e$).
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). Let \(e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).
Dependence Function

Now, let $(E, \mathcal{I}) = (E, r)$ be a matroid, and let $I \in \mathcal{I}$ giving $1_I \in P_r$. Let $e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I)$.

Given $e \in \text{sat}(1_I) \setminus I$ and then consider an $A \ni e$ with $|I \cap A| = r(A)$. 

Note: It might be helpful to include the specific context or example that Prof. Jeff Bilmes was discussing in this part of the lecture.
 Dependence Function

- Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). Let \(e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).
- Given \(e \in \text{sat}(1_I) \setminus I\) and then consider an \(A \ni e\) with \(|I \cap A| = r(A)\).
- Then \(I \cap A\) serves as a base for \(A\) (i.e., \(I \cap A\) spans \(A\)) and any such \(A\) contains a circuit (i.e., we add \(e \notin I\) to \(I \cap A\) w/o increasing rank).
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). Let \(e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

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Given \(e \in \text{sat}(1_I) \setminus I\) and then consider the unique minimal \(A \ni e\) with \(|I \cap A| = r(A)\).
Now, let \((E,I) = (E,r)\) be a matroid, and let \(I \in I\) giving \(1_I \in P_r\). Let \(e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

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Given \(e \in \text{sat}(1_I) \setminus I\) and then consider the unique minimal \(A \ni e\) with \(|I \cap A| = r(A)|\).

That is, consider \(\text{dep}(1_I,e)\), with

\[
\text{dep}(1_I,e) = \bigcap \{A : e \in A \subseteq E, 1_I(A) = r(A)\}\]

\[
= \bigcap \{A : e \in A \subseteq E, |I \cap A| = r(A)\}\]

(18)

(19)
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(l \in \mathcal{I}\) giving \(1_l \in P_r\). Let \(e \in \text{sat}(1_l) = \text{span}(l) = \text{closure}(l)\).

Given \(e \in \text{sat}(1_l) \setminus l\) and then consider an \(A \ni e\) with \(|l \cap A| = r(A)\).

Then \(l \cap A\) serves as a base for \(A\) (i.e., \(l \cap A\) spans \(A\)) and any such \(A\) contains a circuit (i.e., we add \(e \notin l\) to \(l \cap A\) w/o increasing rank).

Given \(e \in \text{sat}(1_l) \setminus l\) and then consider the unique minimal \(A \ni e\) with \(|l \cap A| = r(A)\).

That is, consider \(\text{dep}(1_l, e)\), with

\[
\text{dep}(1_l, e) = \bigcap \{A : e \in A \subseteq E, 1_l(A) = r(A)\} \quad (18)
\]

\[
= \bigcap \{A : e \in A \subseteq E, |l \cap A| = r(A)\} \quad (19)
\]

Then \(\text{dep}(1_l, e)\) must be a circuit since if it included more than a circuit, it would not be minimal in this sense.
Therefore, when \( e \in \text{sat}(1_I) \setminus I \), then \( \text{dep}(1_I, e) = C(I, e) \) where \( C(I, e) \) is the unique circuit contained in \( I + e \) in a matroid (the fundamental circuit of \( e \) in the independent set \( I \) we encountered in Lecture 7).
Dependence Function

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Now, if \( e \in \text{sat}(1_I) \cap I \) with \( I \in \mathcal{I} \), we said that \( C(I, e) \) was undefined (since no circuit is created in this case).
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In this case, for such an $e$, we have $\text{dep}(1_I, e) = \{e\}$ since all such sets $A \ni e$ with $|I \cap A| = r(A)$ contain $e$, but in this case no cycle is created.
**Dependence Function**

- Therefore, when \( e \in \text{sat}(1_I) \setminus l \), then \( \text{dep}(1_I, e) = C(l, e) \) where \( C(l, e) \) is the unique circuit contained in \( l + e \) in a matroid (the **fundamental circuit** of \( e \) in the independent set \( l \) we encountered in Lecture 7).

- Now, if \( e \in \text{sat}(1_I) \cap l \) with \( l \in \mathcal{I} \), we said that \( C(l, e) \) was undefined (since no circuit is created in this case).

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- We are thus free to take subsets of \( l \) as \( A \), all of which must contain \( e \), but all of which have rank equal to size.
Dependence Function

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We are thus free to take subsets of \( I \) as \( A \), all of which must contain \( e \), but all of which have rank equal to size.

Also note: in general for \( x \in P_f \) and \( e \in \text{sat}(x) \), we have \( \text{dep}(x, e) \) is tight by definition.
Dependence Function

- Summarizing,
  - For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\}$
  - For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated ($x$-tight) set w.r.t. $x$ containing $e$. 

Recall, we have $C(I, e) \setminus e' \subseteq I$ for $e' \in C(I, e) \setminus \{e\}$. I.e., $C(I, e)$ consists of the elements that when removed recover independence. In other words, for $e \in \text{span}(I) \setminus I$, we have that $C(I, e) = \{a \in E : I + e - a \notin I\}$ (20). Also, for $e \in \text{span}(I) \setminus I$, we have that $I + e / \notin I$. 

Analogously, for $e \in \text{sat}(x)$, any $x + \alpha 1_e / \notin P_f$ for $\alpha > 0$.

But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$? Could move in this direction if we simultaneously move in another direction.
Dependence Function

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### Dependence Function

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Dependence Function

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**Dependence Function**

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
Dependence Function

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This suggests, for $e \in \text{sat}(x)$, that

$$\text{dep}(x, e) = \{ a : a \in E, \exists \alpha > 0 : x + \alpha(1_e - 1_a) \in P_f \}$$  \hspace{1cm} (21)
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\text{dep}(x, e) = \{ a : a \in E, \exists \alpha > 0 : x + \alpha(1_e - 1_a) \in P_f \} \tag{21}
\]

Viewable in 2D, we have for \( A, B \subseteq E, A \cap B = \emptyset \):

Left: \( A \cap \text{dep}(x, e) = \emptyset \), and we can’t move further in \((e)\) direction by moving in any negative \( a \in A \) direction.

Right: \( A \subseteq \text{dep}(x, e) \), and we can move further in \((e)\) direction by moving in some \( a \in A \) negative direction.
Recall earlier theorem

**Corollary 3.1**

*If $x$ is an extreme point of $P_f$ and $B \subseteq E$ is given such that $\text{supp}(x) \subseteq B \subseteq \text{sat}(x)$, then $x$ is generated using greedy by some ordering of $B$.***
Extreme points by greedy

Recall earlier theorem

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**Proof.**

We know greedy finds an $v \in P_f$ such that for any $c$, $cv = \max(cy : y \in P_f)$, using aforementioned procedure.
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**Proof.**

- We know greedy finds an \( v \in P_f \) such that for any \( c \), \( cv = \max(cy : y \in P_f) \), using aforementioned procedure.
- We also saw that greedy finds an extreme point.
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- We know greedy finds an $v \in P_f$ such that for any $c$, $cv = \max(cy : y \in P_f)$, using aforementioned procedure.
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- A fundamental theorem of convex polytopes: For any convex polytope $P$ in $\mathbb{R}^E$, let $v$ be any vertex of $P$. Then there exists $c \in \mathbb{R}^E$ s.t. $v$ is the unique member of $P$ maximizing $cy$ over $P$. 
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- A fundamental theorem of convex polytopes: For any convex polytope $P$ in $\mathbb{R}^E$, let $v$ be any vertex of $P$. Then there exists $c \in \mathbb{R}^E$ s.t. $v$ is the unique member of $P$ maximizing $cy$ over $P$.
- We then choose $c$ such that $x$ is the desired vertex. This is then achieved by greedy.
We’re next going to study lattices and submodular functions. In doing so, we’ll better be able to understand certain properties of polymatroidal extreme points and ultimately SFM.
A partially ordered set (poset) is a set of objects with an order.
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For any $x, y \in V$, we may ask is $x \leq y$ which is either true or false.

In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):

For all $x$, $x \leq x$.  

If $x \leq y$ and $y \leq x$, then $x = y$. (Reflexive)  

If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity)

If $x \leq y$ and $y \leq x$, then $x = y$. (Antisymmetry)
A partially ordered set (poset) is a set of objects with an order.

Set of objects $V$ and a binary relation $\preceq$ which can be read as “is contained in” or “is part of” or “is less than or equal to”.

For any $x, y \in V$, we may ask is $x \preceq y$ which is either true or false.

In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):

- For all $x$, $x \preceq x$. (Reflexive) (P1.)
- If $x \preceq y$ and $y \preceq x$, then $x = y$. (Antisymmetry) (P2.)
- If $x \preceq y$ and $y \preceq z$, then $x \preceq z$. (Transitivity) (P3.)

We can use the above to get other operators as well such as “less than” via $x \preceq y$ and $x \neq y$ implies $x \prec y$. Also, we get $x \succ y$ if not $x \preceq y$, etc. etc.
There exists only one (unique minimal) element \( x \) which satisfies \( x \preceq y \) for all \( y \). Since if \( x \preceq y \) for all \( y \) and \( z \preceq y \) for all \( y \) then \( z \preceq x \) and \( x \preceq z \) implying \( x = z \). We can name this element 0 (zero). The dual maximal element is called 1.
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Given two elements, we need not have either $x \preceq y$ or $y \preceq x$ be true, i.e., these elements might not be comparable. If for all $x, y \in V$ we have $x \preceq y$ or $y \preceq x$ then the poset is **totally ordered**.
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We define a set of elements \( x_1, x_2, \ldots, x_n \) as a chain if \( x_1 \preceq x_2 \preceq \cdots \preceq x_n \), which means \( x_1 \preceq x_2 \) and \( x_2 \preceq x_3 \) and \( \ldots \) \( x_{n-1} \preceq x_n \). While we normally think of the elements of a chain as distinct they need not be. The length of a chain of \( n \) elements is \( n - 1 \).
Partially ordered set

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Example 4.1

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that $x$ is less than $y$ in the usual sense. Then we have a poset that is actually totally ordered.
Partially ordered set

**Example 4.1**

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that $x$ is less than $y$ in the usual sense. Then we have a poset that is actually totally ordered.

**Example 4.2**

Let $V$ consist of all real single-valued functions $f(x)$ defined on the closed interval $[-1, 1]$, and let $g \preceq f$ mean that $g(x) \leq f(x)$ for all $x \in [-1, 1]$. Again poset, but not total order.
Partially ordered set

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- Any subset of a poset is a poset. If \( S \subseteq V \) than for \( x, y \in S \), \( x \leq y \) is the same as taken from \( V \), but we just restrict the items to \( S \).
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- Any subset of a chain is a chain.
- Two posets $V_1$ and $V_2$ are isomorphic if there is an isomorphism between them (i.e., a 1-1 order preserving (isotone) function that has an order preserving inverse). We write that two posets $U$ and $V$ are isomorphic by $U \simeq V$. 

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**duality.** The dual poset is formed by exchanging $\leq$ with $\geq$. This is called the converse of a partial ordering. The converse of a PO is also a PO. We write the dual of $V$ as $V^d$. $U$ and $V$ are dually isomorphic if $U = V^d$ or equivalently $V = U^d$. When $U = U^d$ then $U$ is self-dual.
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**Example 4.3**

The set $U = 2^E$ for some set $E$ is a poset ordered by set inclusion. See Figure (C). Note that this $U$ is self-dual.
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The set $U = 2^E$ for some set $E$ is a poset ordered by set inclusion. See Figure (C). Note that this $U$ is self-dual.

**Example 4.4**

Given an $n$-dimensional linear (Euclidean) space $\mathbb{R}^n$. A subset of $M \subseteq \mathbb{R}^n$ is an affine set if $(1 - \lambda)x + \lambda y \in M$ whenever $x, y \in M$ and $\lambda \in \mathbb{R}$. A *linear subspace* of $\mathbb{R}^n$ is an affine set that contains the origin. Subspaces can be obtained via some $A, b$ such that for every $y \in M$, $y = Ax + b$ for some $x \in \mathbb{R}^n$.

The set of all linear subspaces of $\mathbb{R}^n$ is a poset (ordered by inclusion), and such a set is self-dual.
Partially ordered set

A chain is saturated if it is a chain of the form

\[ x_1 \prec x_2 \prec \cdots \prec x_n \]

such that \( x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \) (i.e., we have a sequence of coverage relationships where \( x_{i+1} \) covers \( x_i \) for each \( i < n \)).

Hasse-diagram: We can draw a poset using a graph where each \( x \in V \) is a node, and if \( x \sqsubseteq y \) we draw \( y \) directly above \( x \) with a connecting edge, but no other edges.
A partially ordered set (poset) is a set equipped with a binary relation \( \preceq \) that is reflexive, antisymmetric, and transitive. Let \( x \) and \( y \) be elements of a poset. We say \( y \) covers \( x \) if \( x \prec y \) and there exists no \( z \) such that \( x \prec z \prec y \). Note that the inequalities are strict here. We write \( x \sqsubseteq y \) if \( y \) covers \( x \).
Partially ordered set

- **cover** \( y \) covers \( x \) if \( x \prec y \) and there exists no \( z \) such that \( x \prec z \prec y \). Note that the inequalities are strict here. We write \( x \sqsubset y \) if \( y \) covers \( x \).
- A chain is **saturated** if it is a chain of the form \( x_1 \prec x_2 \prec \cdots \prec x_n \) such that \( x_1 \sqsubset x_2 \sqsubset \cdots \sqsubset x_n \) (i.e., we have a sequence of coverage relationships where \( x_{i+1} \) covers \( i \) for each \( i < n \)).
**Partially ordered set**

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- **Hasse-diagram**: We can draw a poset using a graph where each \( x \in V \) is a node, and if \( x \sqsubset y \) we draw \( y \) directly above \( x \) with a connecting edge, but no other edges.
For example, in example (A), we see that \( x \sqsubseteq y \). In example (B) we have \( 3 \preceq 12 \) and \( 3 \sqsubseteq 6 \) but not \( 3 \sqsubseteq 12 \). Hasse diagram for dual order is obtained by turning Hasse diagram upside down.
Partially ordered set

- least element any subset $X \subseteq V$, the least element of $X$ is an element $x \in X$ such that $x \preceq y$ for all $y \in X$. Greatest element is defined similarly.
Partially ordered set

- **least element** any subset $X \subseteq V$, the least element of $X$ is an element $x \in X$ such that $x \preceq y$ for all $y \in X$. Greatest element is defined similarly.

- **minimal element** of a subset $X \subseteq V$ is an element $x \in X$ such that there exists no $y \in X$ such that $y \prec x$. 

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**Theorem 4.5**

*Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.*
Theorem 4.6

Every non-empty finite subset \( X \subseteq V \) has a minimal (and maximal) element.

Proof.

Let \( X = \{x_1, \ldots, x_n\} \). Define \( m_1 = x_1 \) and

\[
m_k = \begin{cases} x_k & \text{if } x_k \prec m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases}
\]  

(22)

Then we have constructed \( m_n \preceq m_{n-1} \preceq \cdots \preceq m_1 \) meaning there is no \( m_k \) for \( k < n \) such that \( m_k \prec m_n \). Let \( M = \{m_1, \ldots, m_n\} \). By construction, we also have that there is no \( x \in X \) with \( x \prec m_n \), thus \( m_n \) is minimal.
In chains elements, minimal equals minimum, and maximal equals maximum.
Partially ordered set

- In chains elements, minimal equals minimum, and maximal equals maximum.
- Given a poset $V$, the length $\ell(V)$ is defined to be the l.u.b. of the lengths of any chains in $V$. That is, $\ell(V)$ is the least upper bound, i.e., smallest number not less than any chain length in $V$. This is finite for finite posets. For example, $\ell(A) = 1$, $\ell(B) = 3$, $\ell(C) = 3$, $\ell(C2) = 4$, $\ell(D) = 2$, $\ell(E) = 2$, $\ell(F) = 3$, and so on.
Partially ordered set

- The **height** or **dimension** of an element $x \in V$, or $l = h(x)$ is the l.u.b. of the lengths of the chains $0 = x_0 < x_1 < \ldots x_l = x$ between 0 and $x$. Note that $h(1) = \ell(V)$ when they exist. $h(x) = 1$ iff $0 \sqsubseteq 1$ and such elements (with unit height) are called “atoms” or “points” or “(ground) elements”.
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If $x \sqsubseteq y$ then $h(x) + 1 = h(y)$.

**graded** posets. Posets may be “graded” by a function $g : V \to \mathbb{Z}$ in the following way:

\[
\begin{align*}
    x \succ y & \Rightarrow g(x) > g(y) \quad \text{(G1)} \\
    x \sqsubseteq y & \Rightarrow g(y) = g(x) + 1 \quad \text{(G2)}
\end{align*}
\]
The height or dimension of an element \( x \in V \), or \( l = h(x) \) is the l.u.b. of the lengths of the chains \( 0 = x_0 < x_1 < \ldots x_l = x \) between 0 and \( x \). Note that \( h(1) = \ell(V) \) when they exist. \( h(x) = 1 \) iff \( 0 \sqsubseteq 1 \) and such elements (with unit height) are called “atoms” or “points” or “(ground) elements”.

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    x \sqsubseteq y & \Rightarrow g(y) = g(x) + 1 \quad \text{(G2)}
\end{align*}
\]

A maximal chain is a chain of unique elements between two elements that cannot be made any longer.
Given two points $x, y \in V$ with $x \succ y$, there might be no or multiple chains between $x$ and $y$. The chains might have different lengths. There might be multiple chains that have the same maximal length.
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**Definition 4.7 (Jordan-Dedekind Chain Condition)**

(or JDCC) All maximal length chains between the same endpoints have the same finite length.
Partially ordered set

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(or JDCC) All maximal length chains between the same endpoints have the same finite length.

**Theorem 4.8**

*Let $V$ be a poset with $0 \in V$ and where all chains are finite. Then $V$ satisfies JDCC iff it is graded by $h(x)$ (the height function).*
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**Proof.**

Grading by $h(x)$ makes JDCC true since the length of any chain between $a \succ b$ is $h(a) - h(b)$. Conversely, given JDCC, and given $h(x)$ as defined (length of the maximal length chain from 0 to $x$), then G1 and G2 follow.
When all maximal length chains between the same endpoints have the same finite length, then we say that the poset is graded by the height. In this case, we say that element $x$ has height or \textit{rank} $h(x)$. The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between $x$ and $y$.

We say a poset is “graded” if it is graded by the height function.
Lattice def

Given $X \subset V$, $y \in V$ is an upper bound of $X$ if $x \leq y$ for all $x \in X$. Note that $y$ need not be in $X$. If $y$ is a least upper bound (l.u.b. $X$ or just sup$X$), then $y \leq z$ for any other upper bound $z$. The l.u.b. if it exists is unique since if $y_1$ and $y_2$ are both l.u.b.’s then $y_1 \leq y_2$ and $y_2 \leq y_1$, or $y_1 = y_2$. Dual definitions for lower bound and greatest lower bound (g.l.b.).
Lattice def

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**Definition 5.1 (lattice)**

A *lattice* is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or “meet” denoted by $x \wedge y \in V$, and also have a l.u.b. or “join” denoted by $x \vee y \in V$. A lattice is “complete” when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).
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Definition 5.1 (lattice)

A lattice is a poset $V$ such that any two elements $x, y \in V$ have a g.l.b. or “meet” denoted by $x \land y \in V$, and also have a l.u.b. or “join” denoted by $x \lor y \in V$. A lattice is “complete” when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of $V$, even of size 1).

- Note again, that such l.u.b.’s and g.l.b.’s are unique if they exist.
Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm \infty$ is complete). $2^E$ for some set $E$ is complete. Note that $E$ can be countably or uncountably infinite.
Lattices

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- All of the figures above are lattices except for (G) and (H). (G) is not a lattice since for example $e \lor f$ does not exist, nor does any join with $e$ exist. (H) is not a lattice since there are two joins for $a$ and $b$. 

\begin{itemize}
  \item[(G)]
    \begin{itemize}
      \item $0$
      \item $1$
      \item $a$
      \item $b$
      \item $c$
      \item $d$
      \item $e$
      \item $f$
    \end{itemize}
  
  \begin{itemize}
    \item $a \preceq b \preceq c$
    \item $d \preceq e \preceq f$
  \end{itemize}

  \begin{itemize}
    \item $a \lor b = c$
    \item $a \lor c = d$
    \item $b \lor c = e$
    \item $d \lor e = f$
  \end{itemize}

\end{itemize}

\begin{itemize}
  \item[(H)]
    \begin{itemize}
      \item $0$
      \item $1$
      \item $a$
      \item $b$
      \item $c$
      \item $d$
    \end{itemize}
  
  \begin{itemize}
    \item $a \preceq b \preceq c$
    \item $d \preceq e \preceq f$
  \end{itemize}

  \begin{itemize}
    \item $a \lor b = c$
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Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a \( \pm \infty \) is complete). \( 2^E \) for some set \( E \) is complete. Note that \( E \) can be countably or uncountably infinite.

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Any non-empty lattice contains a greatest element \( 1 \in V \) and a least element \( 0 \in V \).
Lattices

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- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.
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- Any non-empty lattice contains a greatest element $1 \in V$ and a least element $0 \in V$.

- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.

- In a chain, $x \land y$ is the smaller of the two, and $x \lor y$ is the larger of the two.
Definition 5.2 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.
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A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.

- Given any $x \preceq y$, then all elements $\{ z : x \preceq z \preceq y \}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.
Definition 5.2 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.

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- A convex subset $X$ of a poset $V$ is a subset such that for all $x, y \in V$ with $x \preceq y$, $\{z : x \preceq z \preceq y\} \subseteq X$. A subset $X$ of a lattice $V$ is a convex sublattice if $x, y \in X$ imply that $\{z : x \land y \preceq z \preceq x \lor y\} \subseteq X$. 
A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within $X$ (for all $x, y \in X$, $x \lor y \in X$ and $x \land y \in X$). A sublattice is a lattice.

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- A convex subset $X$ of a poset $V$ is a subset such that for all $x, y \in V$ with $x \preceq y$, $\{z : x \preceq z \preceq y\} \subseteq X$. A subset $X$ of a lattice $V$ is a convex sublattice if $x, y \in X$ imply that $\{z : x \land y \preceq z \preceq x \lor y\} \subseteq X$.

- Obviously, $2^E$ for some set $E$ is a lattice, with join/meet being union/intersection. See Figure(C).
Example 5.3

Given lattices $U, V$, we can form the direct product $UV$ by forming pairs $\{(u, v) : u \in U, v \in V\}$ and ordered so that $(u_1, v_1) \preceq (u_2, v_2)$ iff $u_1 \preceq u_2$ in $U$ and $v_1 \preceq v_2$ in $V$. The direct product of two lattices is a lattice.

Theorem 5.4

In any poset $V$, the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

- $x \land x = x, x \lor x = x$ \hspace{1cm} \text{(Idempotent)} \hspace{1cm} (L1)
- $x \land y = y \land x, x \lor y = y \lor x$ \hspace{1cm} \text{(Commutative)} \hspace{1cm} (L2)
- $x \land (y \land z) = (x \land y) \land z, x \lor (y \lor z) = (x \lor y) \lor z$ \hspace{1cm} \text{(Associative)} \hspace{1cm} (L3)
- $x \land (x \lor y) = x \lor (x \land y) = x$ \hspace{1cm} \text{(Absorption)} \hspace{1cm} (L4)
- $x \preceq y \iff x \land y = x \text{ and } x \lor y = y$ \hspace{1cm} \text{(Consistency)} \hspace{1cm} (CON)
Lattices

Example 5.3

Given lattices $U$, $V$, we can form the direct product $UV$ by forming pairs 
$\{(u, v) : u \in U, v \in V\}$ and ordered so that $(u_1, v_1) \preceq (u_2, v_2)$ iff $u_1 \leq u_2$ in $U$ and $v_1 \leq v_2$ in $V$. The direct product of two lattices is a lattice.

Theorem 5.4

In any poset $V$, the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

- $x \land x = x, x \lor x = x$ (Idempotent) (L1)
- $x \land y = y \land x, x \lor y = y \lor x$ (Commutative) (L2)
- $x \land (y \land z) = (x \land y) \land z, x \lor (y \lor z) = (x \lor y) \lor z$ (Associative) (L3)
- $x \land (x \lor y) = x \lor (x \land y) = x$ (Absorption) (L4)
- $x \preceq y \iff x \land y = x$ and $x \lor y = y$ (Consistency) (CON)

Note the above works for posets, not necessary for it to be a lattice.
Theorem 5.5

*Given a poset $V$ with $0 \in V$, then for all $x \in V$,*

$$0 \land x = 0 \text{ and } 0 \lor x = x$$  \hfill (23)
Theorem 5.5

*Given a poset \( V \) with \( 0 \in V \), then for all \( x \in V \),
\[
0 \land x = 0 \quad \text{and} \quad 0 \lor x = x
\]  
(23)

Theorem 5.6

*In any lattice, the operations of join and meet are order-preserving in the following sense:
\[
y \preceq z \Rightarrow x \land y \preceq x \land z \quad \text{and} \quad x \lor y \preceq x \lor z
\]  
(24)
Lattices

Theorem 5.5

Given a poset $V$ with $0 \in V$, then for all $x \in V$,

\[ 0 \land x = 0 \quad \text{and} \quad 0 \lor x = x \]  

(23)

Theorem 5.6

In any lattice, the operations of join and meet are order-preserving in the following sense:

\[ y \preceq z \Rightarrow x \land y \preceq x \land z \quad \text{and} \quad x \lor y \preceq x \lor z \]  

(24)

Theorem 5.7

In any lattice, the following distributive inequalities hold for all $x, y, z \in V$:

\[ x \land (y \lor z) \succeq (x \land y) \lor (x \land z) \]  

(25a)

\[ x \lor (y \land z) \preceq (x \lor y) \land (x \lor z) \]  

(25b)
Theorem 5.8

In any lattice, the following distributive inequalities hold for all \( x, y, z \in V \):

\[
\begin{align*}
x \land (y \lor z) &\leq (x \land y) \lor (x \land z) \\
x \lor (y \land z) &\leq (x \lor y) \land (x \lor z)
\end{align*}
\] (26a) (26b)

Proof.

We have \( x \land y \leq x \) and \( x \land y \leq y \leq y \lor z \). Therefore, \( x \land y \leq x \land (y \lor z) \).

Also, since \( x \land z \leq x \) and \( x \land z \leq z \leq y \lor z \), we have \( x \land z \leq x \land (y \lor z) \).

Thus, \( x \land (y \lor z) \) is an upper bound of both \( x \land y \) and \( x \land z \), which means that it is an upper bound of their join.

Eq 26b is by duality.
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

For example, in (D), we have that
\[ a \land (b \lor c) = a \land 1 = a \]
but
\[ (a \land b) \lor (a \land c) = 0 \lor 0 = 0 \]
and obviously
\[ a \succ 0. \]
Also, in (D) we have
\[ a \lor (b \land c) = a \lor 0 = a \preceq (a \lor b) \land (a \lor c) = 1 \land 1 = 1. \]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

- For example, in (D), we have that $a \land (b \lor c) = a \land 1 = a$ but $(a \land b) \lor (a \land c) = 0 \lor 0 = 0$ and obviously $a \succ 0$. 
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

For example, in (D), we have that \( a \land (b \lor c) = a \land 1 = a \) but \( (a \land b) \lor (a \land c) = 0 \lor 0 = 0 \) and obviously \( a \succ 0 \).

Also, in (D) we have \( a \lor (b \land c) = a \lor 0 = a \prec (a \lor b) \land (a \lor c) = 1 \land 1 = 1 \).
Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

\[(E)\quad \begin{align*}
1 \\
\bullet \\
\downarrow \\
a \\
\bullet \\
\downarrow \\
0
\end{align*}\]
Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

\[ c \land (a \lor b) = c \land 1 = c \succ\]
\[ (c \land a) \lor (c \land b) = 0 \lor b = b \]
Distributive Inequalities

Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).

In (E), we have that
\[ c \land (a \lor b) = c \land 1 = c \succeq \]
\[ (c \land a) \lor (c \land b) = 0 \lor b = b \]

Also, in (E), we have
\[ b \lor (a \land c)b \lor 0 = b \prec \]
\[ (b \lor a) \land (b \lor c) = 1 \land c = c \]
Modular inequality

**Theorem 5.9**

In any lattice, the following modular inequalities holds for all \( x, y, z \in V \):

\[
x \preceq z \implies x \lor (x \land z) \preceq (x \lor y) \land z
\]

(27)
Theorem 5.9

In any lattice, the following modular inequalities holds for all \( x, y, z \in V \):
\[
x \leq z \Rightarrow x \lor (x \land z) \leq (x \lor y) \land z
\]  
(27)

Proof.

We have that \( x \leq x \lor y \) and given \( x \leq z \). Then, \( x \leq (x \lor y) \land z \).

Also, \( y \land z \leq y \leq x \lor y \) and with \( y \lor z \leq z \) gives \( y \land z \leq (x \lor y) \land z \).

Then \( x \lor (y \land z) \leq (x \lor y) \land z \).
Modular inequality

**Theorem 5.9**

In any lattice, the following modular inequalities holds for all \( x, y, z \in V \):

\[
x \leq z \Rightarrow x \lor (x \land z) \leq (x \lor y) \land z
\]

(27)

**Proof.**

We have that \( x \leq x \lor y \) and given \( x \leq z \). Then, \( x \leq (x \lor y) \land z \).

Also, \( y \land z \leq y \leq x \lor y \) and with \( y \lor z \leq z \) gives \( y \land z \leq (x \lor y) \land z \).

Then \( x \lor (y \land z) \leq (x \lor y) \land z \).

- The term “modular” somehow comes from abstract algebra, where a R-module is an abstract system that generalizes \((\mathbb{R}, \mathbb{R}^n)\) (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this property.
Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.
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Theorem 5.10

In any lattice, the following are equivalent:

- \( x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \) (28a)
- \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \) (28b)
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\begin{align*}
&\text{Theorem 5.10} \\
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&\quad x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \quad (28b)
\end{align*}

It is important to note the $\forall x, y, z$ since this is not true only for individual elements. Note moreover that this means that the operators $\lor = +$ and $\land = \cdot$ do not form a lattice over $\mathbb{R}$.
Distributive Lattices

Theorem 5.11

In any lattice, the following are equivalent:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \]  
\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \forall x, y, z \]  

(29a)  
(29b)
Distributive Lattices

Theorem 5.11

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\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \forall x, y, z \quad (29a) \]
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Proof.

Take as given the 2nd equation and show the first. Then

\[ (x \land y) \lor (x \land z) = [(x \land y) \lor x] \land [(x \land y) \lor z] \]
\[ = x \land [(x \land y) \lor z] \]
\[ = x \land [(x \lor z) \land (y \lor z)] \]
\[ = x \land (x \lor z) \land (y \lor z) \quad \text{associative} \]
\[ = x \land (y \lor z) \]

by the 2nd eq

\[ x \land y \leq x \quad (31) \]
\[ x \lor z \geq x \quad (34) \]
Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
Distributive Lattices

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- Thus a lattice is distributive if either of the above equalities hold.
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- Thus a lattice is distributive if either of the above equalities hold.

**Example 5.12**

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \leq y$ mean that $x$ divides $y$. I.e., $2 \leq 4$ but $2 \not\leq 5$. Then this is lattice with $x \lor y = \text{l.c.m.}(x, y)$ and $x \land y = \text{g.c.d.}(x, y)$. It is also distributive. Again consider figure (B).
Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.
- Thus a lattice is distributive if either of the above equalities hold.

**Example 5.12**

Let \( V = \mathbb{Z}^+ \) be the set of positive integers and let \( x \preceq y \) mean that \( x \) divides \( y \). I.e., \( 2 \preceq 4 \) but \( 2 \npreceq 5 \). Then this is lattice with \( x \lor y = \text{l.c.m.}(x, y) \) and \( x \land y = \text{g.c.d.}(x, y) \). It is also distributive. Again consider figure (B).

**Theorem 5.13 (identity)**

In a distributive lattice, if \( z \land x = z \land y \) and \( z \lor x = z \lor y \) then \( x = y \).
In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.
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**Definition 6.1 (modular identity)**

If $x \leq z$, then $x \lor (y \land z) = (x \lor y) \land z$.  

(L5)
In the above we also defined the modular inequality. We can strengthen this as well to get what is known as the modular identity.

**Definition 6.1 (modular identity)**

\[ x \leq z, \text{ then } x \lor (y \land z) = (x \lor y) \land z. \]  \hspace{1cm} (L5)

Clearly any distributive lattice satisfies the modular identity since when \( x \leq z \) we have that \( x \lor z = z \) and this gives us the 2nd of the distributive lattice equalities above.
Not every lattice is modular. Figure (D) is modular but not distributive. We already saw that (D) is not distributive since it is strict for certain assignments. It is modular though.
Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \lor (a \land c) = b \lor 0 = b \prec (b \lor a) \land c = 1 \land c = c$, so modular equality is violated.
Modular Lattices

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**Theorem 6.2**

*Any non-modular lattice $V$ contains the lattice in Figure (E) as a sublattice.*
Modular Lattices

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  \]
  so modular equality is violated.

**Theorem 6.2**

Any non-modular lattice \( V \) contains the lattice in Figure (E) as a sublattice.

- Thus, the structure (E) is fundamental to non-modular lattices.
Modular Lattices

Theorem 6.3

A necessary and sufficient condition for a modular lattice is to have both:

Upper-Semimodularity if $x$ and $y$ cover $z$ and $x \neq y$ then $x \lor y$ covers both $x$ and $y$, and

Lower-Semimodularity if $z$ covers $x$ and $y$ and $x \neq y$ then $x$ and $y$ both covers $x \land y$.

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
Modular Lattices

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\[
\begin{align*}
x \lor y = z \\
x \land y = x \lor y
\end{align*}
\]

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\[
\begin{align*}
\text{Modular Lattices} & \quad \text{Theorem 6.3} \\
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\text{Lower-Semimodularity if } z \text{ covers } x & \text{ and } y \text{ and } x \neq y \text{ then } x \text{ and } y \text{ both cover } x \land y.
\end{align*}
\]
Modular Lattices

**Theorem 6.3**

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Thus, upper-semimodularity means that if $z \sqsubseteq x$ and $z \sqsubseteq y$, and if $x \neq y$, then $x \sqsubseteq (x \lor y)$ and $y \sqsubseteq (x \lor y)$.
Modular Lattices

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Thus, upper-semimodularity means that if $z \sqsubseteq x$ and $z \sqsubseteq y$, and if $x \neq y$, then $x \sqsubseteq (x \lor y)$ and $y \sqsubseteq (x \lor y)$.

Thus, lower-semimodularity means that if $x \sqsubseteq z$ and $y \sqsubseteq z$, and if $x \neq y$, then $(x \land y) \sqsubseteq x$ and $(x \land y) \sqsubseteq y$. 

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Modular Lattices

Theorem 6.3

A necessary and sufficient condition for a modular lattice is to have both:

**Upper-Semimodularity** if \( x \) and \( y \) cover \( z \) and \( x \neq y \) then \( x \lor y \) covers both \( x \) and \( y \), and

\[
\begin{align*}
x & \quad \Rightarrow \\
y & \\
z & \\
x \lor y & \\
\end{align*}
\]

**Lower-Semimodularity** if \( z \) covers \( x \) and \( y \) and \( x \neq y \) then \( x \) and \( y \) both cover \( x \land y \).

\[
\begin{align*}
x & \\
y & \\
z & \Rightarrow \\
x \land y & \\
\end{align*}
\]

As we will see, the first equation implies submodularity on the dimension (height function) and the second equation implies supermodularity on the dimension (height) function. Both together imply modularity on the dimension function.
**Theorem 7.1**

Let $L$ be a finite lattice. The following two conditions are equivalent:

(i) $L$ is graded, and the height function $h(\cdot)$ of $L$ satisfies the (what we know as the submodular) inequality for all $x, y \in L$.

$$h(x) + h(y) \geq h(x \lor y) + h(x \land y)$$  \hspace{1cm} (35)

(ii) If $x$ and $y$ both cover $z$, then $x \lor y$ covers both $x$ and $y$
Proof.

(i) ⇒ (ii).
Suppose $x$ and $y$ cover $z$. Note that if $x$ and $y$ cover $z$ then since $L$ is a lattice, $z = x \land y$. Then we have $h(x) = h(y) = h(x \land y) + 1$. Also, since $x$ and $y$ are distinct, and since they both cover $z$ we can’t have (w.l.o.g.) $x \leq y$, and thus $h(x \lor y) > h(x) = h(y)$. Hence by (i), we have

\[ h(x) + h(y) - h(x \land y) \geq h(x \lor y) > h(x \land y) + 1 \quad (36) \]

or

\[ h(x \land y) + 2 \geq h(x \lor y) > h(x \land y) + 1 \quad (37) \]

giving $h(x \lor y) = h(x \land y) + 2 = h(x) + 1 = h(y) + 1$, so that $x \lor y$ covers both $x$ and $y$.

(ii) ⇒ (i). Suppose $L$ is not graded, and let $[u, v]$ be an interval of $L$ of minimal length that is not graded. Then there are elements $x_1, x_2$ of $[u, v]$ that cover $u$ and such that all maximal chains of each interval $[x_i, v]$ have the same length $\ell_i$ where $\ell_1 \neq \ell_2$. By (ii), there saturated chains in $[x_i, v]$ of the form $x_i \prec x_1 \lor x_2 \prec y_1 \prec y_2 \prec \cdots \prec y_k = v$,
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.
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\[
\begin{array}{c}
x \vee y \\
x \\
x \wedge y
\end{array}
\]

\[
\begin{array}{c}
x \\
x \\
x \wedge y
\end{array}
\]

\[
\begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array}
\]

\[
2 + 2 > 3 + 0
\]
Submodular Lattices

The next figure is an example of an upper-semimodular (or a “submodular”) lattice over 7 elements.

Flip it up side down to get a lower-semimodular (or “supermodular”) lattice.

2 + 2 > 3 + 0
### Lattices

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Lattices are structures that model the relationships between elements of a set. They are often used in various fields such as computer science, mathematics, and engineering. The study of lattices includes topics such as modular lattices and submodular lattices.

**Modular Lattices**

Modular lattices are a special type of lattice where the distributive law holds for any two elements. They are named after the mathematician Garrett Birkhoff, who made significant contributions to the theory of lattices.

**Submodular Lattices**

Submodular functions are a type of function where adding an element to a set does not increase the function value by more than a linear amount. Submodular lattices are a generalization of submodular functions to the lattice setting.

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**Summary**

In summary, lattices provide a framework for understanding and modeling the relationships between elements in a set. They are fundamental in various areas of mathematics and have applications in computer science and engineering.
Lattices
Sources for Today’s Lecture