EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 12 - May 11th, 2011

Announcements

- On Final projects. **One** single page final project updates due next Wednesday, 5/18 at 5:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.

Class Road Map

We need to find one makeup lectures this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11): Lovász ext. + polymatroid props.
- L13 (5/13):
- L14 (5/18):
- L15 (5/20):
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/?): (need to find time/date/place).

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  3. This works only for the difference between r and x, but we'd like an algorithm that works for any arbitrary submodular function f, even non-monotone and/or non-non-increasing/decreasing.
- It turns out that (2) and (3) are easy to deal with, but (1) took another 16 years to solve. In fact, the problem can still be seen as unsolved, if we want a reasonable, scalable, guaranteed low-order polynomial algorithm.

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- This is true iff $0 \leq f(A) - x(A), \forall A \subseteq E$.
- And this is true iff $\min f(A) - x(A) \geq 0$.
- So, given a strongly polynomial time algorithm for general submodular function minimization, we can test polyhedral membership, in at least this limited (polymatroidal polytope) sense.

Polymatroidal polyhedron and greedy

- We have a result very similar to that of matroids.

Theorem 2.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

An extension of f

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- That is, we have, for submodular f,
$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{2}$$
$$= \sum_{i=1}^{m} w(e_i)(f(U_i) - f(U_{i-1})) \tag{3}$$
$$= w(e_m)f(U_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(U_i) \tag{4}$$
where $U_i = \{e_1, e_2, \ldots, e_i\}$ based on the elements of E being named, w.l.o.g., in order of decreasing w, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

An extension of f

$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{5}$$

- Therefore, if f is a submodular function, we can write

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(U_i) \tag{6}$$

where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted according to w as before.

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where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted according to w as before.
- Clearly, $\tilde{f}(w) = \max(wx : x \in P_f)$ is always convex in w, since it is the maximum of a set of linear functions (even when f is not submodular, or P_f is an arbitrary set).

An extension of f

- On the other hand, for any f (even not submodular), we can define an extension in this way, with

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with the U_i's and sorted order of w defined as above, so that $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{U_i}$

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- This "extension" of f, in any case, is called the Lovász extension of f.

Lovász Extension, submodularity and convexity

Theorem 3.1

A function $f : 2^E \to \mathbb{R}$ is submodular iff its Lovász extension $\tilde{f}$ of f is convex.

Proof.

seen in last lecture. □

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- Edmonds-Lovász:
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- Choquet?

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- Lovász-Edmonds:
- Choquet?
- Choquet in 1955 defined what is now known as the Choquet integral, which is a form of "non-linear" integration over discrete finite sets. This turns out to be equivalent to the Lovász extension, as we next see.

Integration

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- Lebesgue integration allows integration w.r.t. an underlying measure μ of sets. E.g., given measurable function f, we can define
$$\int_X f du = \sup I_X(s) \tag{8}$$
where $I_X(s) = \sum_{i=1}^n c_i \mu(X \cap X_i)$, and where we take the sup over all measurable functions s such that $0 \leq s \leq f$ and $s(x) = \sum_{i=1}^n c_i I_{X_i}(x)$ and where $I_{X_i}(x)$ is indicator of membership of set X_i, with $c_i > 0$.

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$$\text{WAVG}(x) = \sum_{e \in E} x(e) w(e) \tag{9}$$

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- Consider $\mathbf{1}_e$ for $e \in E$, we have

$$\text{WAVG}(\mathbf{1}_e) = w(e) \tag{10}$$

so seen as a function on the hypercube vertices, the entire WAVG function is given based on values on a subset of the vertices of this hypercube, i.e., $\{\mathbf{1}_e : e \in E\}$. Moreover, we are interpolating as in

$$\text{WAVG}(x) = \sum_{e \in E} x(e) w(e) = \sum_{e \in E} x(e) \text{WAVG}(\mathbf{1}_e) \tag{11}$$

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- More complex aggregation functions can be constructed by defining the aggregation function on all vertices of the hypercube. I.e., for each $\mathbf{1}_A : A \subseteq E$ we might have (for all $A \subseteq E$):

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- What then might $\text{AG}(x)$ be for some $x \in \mathbb{R}^E$? Our weighted average functions might look something more like the r.h.s. in:
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Of course, this alone doesn't do it as we can define $w'(e) = \sum(w_A : e \in A)$ and get back a normal weighted average.

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- Set function $f : 2^E \to \mathbb{R}$ is a game if f is normalized $f(\emptyset) = 0$.

Integration

- Set function $f : 2^E \to \mathbb{R}$ is called a capacity if it is monotone non-decreasing, i.e., $f(A) \leq f(B)$ whenever $A \subseteq B$.

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- Also, If we have an expression for f_b we can construct a set function f as $f(A) = f_b(\mathbf{1}_A)$. We can also often relax f_b to any $x \in [0,1]^m$.

Choquet integral

Definition 4.1

Let f be any capacity on E and $w \in \mathbb{R}_+^E$. The Choquet integral (1954) of w w.r.t. f is defined by

$$C_f(w) = \sum_{i=1}^{m} (w_{e_i} - w_{e_{i+1}}) f(U_i) \tag{14}$$

where in the sum, we have sorted and renamed the elements of E so that $w_{e_1} \geq w_{e_2} \geq \dots \geq w_{e_m} \geq w_{e_{m+1}} = 0$, and where $U_i = \{e_1, e_2, \dots, e_i\}$.

- We immediately see that an equivalent formula is as follows:

$$C_f(w) = \sum_{i=1}^{m} w(e_i)(f(U_i) - f(U_{i-1})) \tag{15}$$

where $U_0 \stackrel{\text{def}}{=} \emptyset$.

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- BTW: this essentially Abel's partial summation formula: Given two arbitrary sequences $\{a_n\}$ and $\{b_n\}$ with $A_n = \sum_{k=1}^n a_k$, we have

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1} - A_{m-1} b_m \tag{16}$$

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- A function can be defined on a segment of $\mathbb{R}$, namely $w_{e_i} > \alpha \geq w_{e_{i+1}}$. This function $F_i : [w_{e_{i+1}}, w_{e_i}) \to \mathbb{R}$ is defined as

$$F_i(\alpha) = f(\{e \in E : w_e > \alpha\}) = f(U_i) \tag{17}$$

Choquet integral

- We can generalize this to multiple segments of $\mathbb{R}$ (for now, take $w \in \mathbb{R}_+^E$). I.e.,

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha > w_1 \\ f(\{e \in E : w_e > \alpha\}) & \text{if } w_{e_i} > \alpha \geq w_{e_{i+1}} \\ 0 & \text{if } w_m \geq \alpha \geq 0 \end{cases} \quad (18)$$

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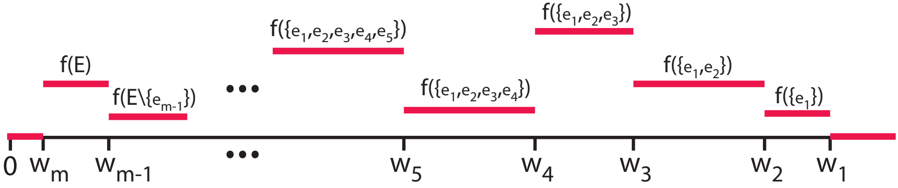
- Visualizing this, we see that we've got a piecewise constant function, where the constant values are given by f evaluated on U_i for each i



Note, depected may be a game but not a capacity.

Choquet integral

- Now consider the integral, with $w \in \mathbb{R}_+^E$, and normalized f so that $f(\emptyset) = 0$. Recall $w_{m+1} \stackrel{\text{def}}{=} 0$.

$$\tilde{f}(w) \stackrel{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \tag{19}$$

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$$= \sum_{i=1}^m \int_{w_{i+1}}^{w_i} f(U_i) d\alpha = \sum_{i=1}^m f(U_i)(w_i - w_{i+1}) \tag{23}$$

Choquet integral

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Given $w \in \mathbb{R}_+^E$, the Lovász extension (equivalently Choquet integral) may be defined as follows:

$$\tilde{f}(w) \stackrel{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \tag{24}$$

where the function F is defined as before.

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where the function F is defined as before.

- Note that it is not necessary in general to require $w \in \mathbb{R}_+^E$ (i.e., we can take $w \in \mathbb{R}^E$) nor that f be non-negative, but it is a bit more involved. Above is the simple case.

Lovász extension

- For a given $w \in [0,1]^m$, it is easy to see that we can also define the Lovász extension as
$$\tilde{f}(w) = \mathbb{E}[f(e \in E : w(e_i) > \alpha)] \tag{25}$$
where α is uniform random variable in $[0,1]$.

Lovász extension

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$$\tilde{f}(w) = \mathbb{E}[f(e \in E : w(e_i) > \alpha)] \tag{25}$$
where α is uniform random variable in $[0,1]$.
- The convexity of the Lovász extension, the ease of minimizing convex functions, and the fact that we can recover f from $\tilde{f}$ via $f(A) = \tilde{f}(\mathbf{1}_A)$ corresponds to why SFM is possible in polynomial time (which was first shown by Grötschel, Lovász, and Schrijver in 1988 as part of their Ellipsoid method.

Choquet integral and aggregation

- Recall, we want to produce some notion of generalized aggregation function having the flavor of:

$$\text{AG}(x) = \sum_{A \subseteq E} x(A) w_A = \sum_{A \subseteq E} x(A) \text{AG}(\mathbf{1}_A) \tag{26}$$

how does this correspond to Lovász extension?

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- Let us partition the hypercube $[0,1]^m$ into q polytopes defined by a set of vertices $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_q$. This forms a "triangulation" of the hypercube.

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- For any $x \in [0,1]^m$ there is a $\mathcal{V}(x) = \mathcal{V}_j$ for some j such that $x \in \mathrm{conv}(\mathcal{V}(x))$.

Choquet integral and aggregation

- For $x \in [0,1]^m$, let us define the (unique) coefficients $\alpha_0^x(A)$ and $\alpha_i^x(A)$ so that x can be represented as a weighted combination of elements in $\mathcal{V}(x)$. Note that many of these coefficient might be zero.

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- From this, we can define an aggregation function of the form

$$\text{AG}(x) \stackrel{\text{def}}{=} \sum_{A:\mathbf{1}_A \in \mathcal{V}(x)} \left(\alpha_0^x(A) + \sum_{i=1}^m \alpha_i^x(A) x_i \right) \text{AG}(\mathbf{1}_A) \tag{27}$$

Choquet integral and aggregation

- We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation σ, define

$$\text{conv}(\mathcal{V}_\sigma) = \left\{ x \in [0,1]^n | x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(m)} \right\} \quad (28)$$

Then these $m!$ blocks of the partition are called the canonical partitions of the hypercube. In this case, we have:

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Proposition 4.3

The above linear interpolation using the canonical partition yields the Lovász extension.

Choquet integral and aggregation

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- The greedy algorithm does more than solve $\max(wx : x \in P_f)$. We can use it to generate vertices of polymatroidal polytopes.

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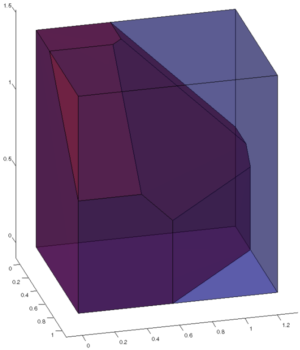

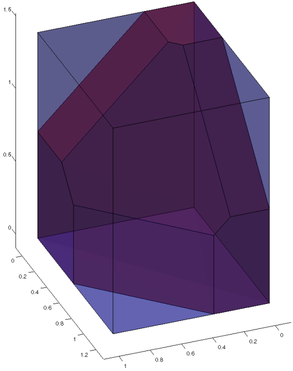

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- The base polytope is defined as the extreme face of P_f. I.e.,

$$B_f = P_f \cap \left\{ x \in \mathbb{R}_+^E : x(E) = f(E) \right\} \tag{29}$$

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- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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$$x(e_1) = f(E_1) \tag{30}$$

$$x(e_j) = f(E_j) - f(E_{j-1}) \text{ for } 2 \leq j \leq i \tag{31}$$

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 5.1

For a given ordering $E = (e_1, \ldots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).

□

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Proof.

- We already saw that $x \in P_f$ (in Lecture 11, proof of Theorem 4.2).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

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- And so on ...
- Also, since $x \in P_f$, for each i, we see that,

$$x(E_i) = f(E_i) \tag{35}$$

$$x(A) \leq f(A), \forall A \subseteq E \tag{36}$$

Polymatroid extreme points

- As an example, we have $x(E_1) = x(e_1) = f(e_1)$
- $x(E_2) = x(e_1) + x(e_2) = f(e_1, e_2)$ so $x(e_2) = f(e_1, e_2) - x(e_1) = f(e_1, e_2) - f(e_1)$.
- $x(E_3) = x(e_1) + x(e_2) + x(e_3) = f(e_1, e_2, e_3)$ so $x(e_3) = f(e_1, e_2, e_3) - x(e_2) - x(e_1) = f(e_1, e_2, e_3) - f(e_1, e_2)$
- And so on ...
- Also, since $x \in P_f$, for each i, we see that,

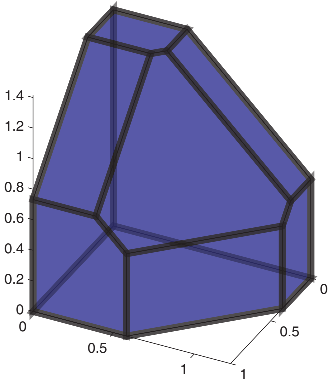
$$x(E_i) = f(E_i) \quad (35)$$

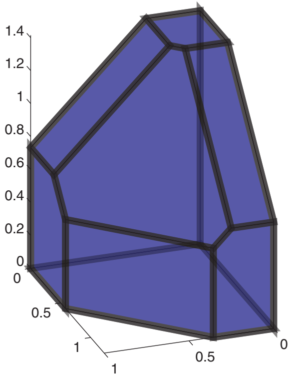
$$x(A) \leq f(A), \forall A \subseteq E \quad (36)$$

- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order.

Polymatroid extreme points

some examples





Polymatroid extreme points

- Moreover, we have

Corollary 5.2

If x is an extreme point of P_f and $A \subseteq E$ is given such that $\{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A))$, then x is generated using greedy by some ordering of A.

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- Note, $\text{cl}(x) = \cup(A : x(A) = f(A))$ is the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, see Lecture 7, in proof of Theorem 4.3, starting Eq. 50).

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Polymatroid extreme points

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- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\text{supp}(x)$ is not tight, but for an extreme point, $\text{supp}(x)$ is.

Scratch Paper

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Sources for Today's Lecture

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- L. Schrijver, "Combinatorial Optimization", 2003.
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