EE595A – Submodular functions, their optimization and applications – Spring 2011

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 11 - May 6th, 2011

Announcements

- On Final projects. **One** single page final project proposals (revision one) are due today at 6:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.
- Ideal proposal would, say, lead to a NIPS paper in June and be related to submodularity.

Class Road Map

We need to find one makeup lectures this term.

- L1 (3/30):
- L2 (4/1):
- L3 (4/6):
- L4 (4/8):
- L5 (4/13):
- L6 (4/15):
- L7 (4/20):
- L8 (4/27):
- L9 (4/29):
- L10 (5/4):
- L11 (5/6): On SFM, polymatroid member & greedy, Lovász ext.
- L12 (5/11):
- L13 (5/13):
- L14 (5/18):
- L15 (5/20):
- L16 (5/25):
- L17 (5/27):
- L18 (6/1):
- L19 (6/3):
- L20: (6/?): (need to find time/date/place).

Most violated inequality problem

- Consider
$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \tag{1}$$
- We saw before that $P_r = P_{\text{ind. set}}$.
- Suppose we have any $x \in \mathbb{R}^E_+$ such that $x \notin P_r$.
- The most violated inequality when x is considered w.r.t. P_r corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., $\max \{x(A) - r_M(A) : A \subseteq E\}$.
- This corresponds to $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min \{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

Augmenting path theorem

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Theorem 2.1

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Augmenting path theorem consequences

Corollary 2.2

For any $x \in \mathbb{R}_+^E$*, we have*

$$\max\left(y(E) : y \leq x, y \in P_r\right) = \min\left(x(A) + r(E \setminus A) : A \subseteq E\right) \quad (2)$$

Note: this was not used in the theorem above, rather it is a consequence!

Proof.

1. First, as we've seen, any $y \in P$ with $y \leq x$, and any $A \subset E$, we have

$$y(E) = y(A) + y(E \setminus A) \leq r(A) + x(E \setminus A) \quad (3)$$

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3. Choose any $y \in P$ such that $y \leq x$ and with $y(E)$ maximum and run algorithm.
4. Then eventually exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).

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4. Then eventually exists no such $y' \in P$ s.t. $y'(E) > y(E)$, and the digraph won't have a directed path from s to t (by the theorem).
5. Then, there is a set A such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, giving $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality in Eq. 3, and minimality of $r(A) + x(E \setminus A)$.

=> minimizes $r(A) - x(A)$

Augmenting path theorem consequences

Corollary 2.3

Given matroid M, we have

$$P_{ind.\ set} = P_r \tag{4}$$

We even get this a consequence!

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Let $G_0, G_1, \ldots, G_k$ be a sequence of digraphs, each having vertex set $E \cup \{s, t\}$, and correspond to such graphs each one running the algorithm implied by theorem 4.1 Assume fixed total order of $E \cup \{s\}$. Let Q_i denote the CBFS path in G_i, for $0 \leq i < k$. If it is the case that, for $0 \leq i < k$:

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2. *If (e, f) is an edge in G_{i+1} but not in G_i, then $e, f \in E$ and there are vertices $a, b \in Q_i$ with a preceding b on Q_i such that: 1) either $a = f$ or (a, f) is an edge in G_i; and 2) $b = e$ or (e, b) is an edge in G_i,*

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Then we have that the number of augmentations has bound $k \leq |E|^3$.

Bounding the number of augmenting paths

Theorem 2.5

It is possible to construct an augmentation scheme such that each augmenting path is done in accordance to Theorem 4.4. Each such augmentation is CBFS, and is called a "grand" augmentation, and is maximal in a certain way. This achieves the $O(n^3)$ time, in the number of augmentations, mentioned above.

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- Of course, the cost of each augmentation might be expensive. For matric matroids, each would be $O(r^2 n^5)$ where r is the number of rows of the matrix, leading to $O(r^2 n^8)$ algorithm.

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- On the other hand, this algorithm has some intriguing properties.

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- Recall, we are given $x \in \mathcal{R}_+^E$. Algorithm implied by this theorem is called multiple times, setting $y \leftarrow y'$, until no such path exists at which point we get said A and y s.t. $y \leq x$ and y is otherwise maximal in P.

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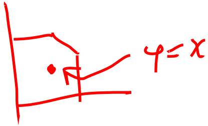

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- If $x \notin P$, minimizing $r(A) - x(A)$ gives an A so that gives the inequality, of the form $x(A) \leq r(A)$ that is most violated and $E \setminus A = \{e \in E : x(e) > y(e)\}$.

Jack Edmonds and Eugene Lawler, 1977, Banff

[Edmonds]
"But now, you know, this is my day in the sun." - from A Glimpse of Heaven, *1991.*

Towards SFM

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- It turns out that (2) and (3) is easy to deal with, but (1) took another 16 years to solve (and perhaps can still be seen as unsolved, w.r.t. wanting a scalable algorithm).

Addressing Monotonicity

- First, given any submodular function g, construct modular function $m : E \to \mathbb{R}$ such that $m(e) = g(E \setminus \{e\}) - g(E)$.

$= -[g(E) - g(E \setminus e)]$

$= -$[gain of adding e to $E \setminus e$]

$= -$[smallest possible addition value of e in some context].

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- Then $f(\emptyset) = 0$, so f is normalized.
- Also, f is monotone non-decreasing (and thus non-negative) and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since for $v \notin B$,

$$f(B + v) - f(B) = g(B + v) - g(B) + m(v) \tag{6}$$

$$= g(B + v) - g(B) + g(E - v) - g(E) \tag{7}$$

$$\geq 0 \tag{8}$$

since, by submodularity, $g(B + v) - g(B) \geq g(E) - g(E - v)$. (9)

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, g is arbitrary submodular.

- Also, if we wish to minimize g, then given $f(A) = g(A) + m(A) - g(\emptyset)$, we can just minimize $f - m$ since $g(\emptyset)$ is a constant. $f - m = g - g(\emptyset)$

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- No, but for any e such that $m(e) < 0$, e can't be a minimizer of $f - m$ since, assuming that A minimizes $f(A) - m(A)$ and $e \in A$ is such that $m(e) < 0$, then we have that $f(A') - m(A') < f(A) - m(A)$ where $A' = A \setminus \{e\}$.

$f(A') \leq f(A)$

$-m(A') < -m(A)$

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- This follows since f is monotone non-decreasing, and $m(A) = m(A') + m(e)$, so $m(A') > m(A)$.

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- No, but for any e such that $m(e) < 0$, e can't be a minimizer of $f - m$ since, assuming that A minimizes $f(A) - m(A)$ and $e \in A$ is such that $m(e) < 0$, then we have that $f(A') - m(A') < f(A) - m(A)$ where $A' = A \setminus \{e\}$.
- This follows since f is monotone non-decreasing, and $m(A) = m(A') + m(e)$, so $m(A') > m(A)$.
- So we "throw away" any e s.t. $m(e) < 0$. This deals with (2) above.

$E' = E \setminus M \quad M = \{e : m(e) < 0\}$

$\Rightarrow f' : 2^{E'} \to \mathbb{R}, \quad f'(A) = f(A)$ for $A \subseteq E'$

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- So we "throw away" any e s.t. $m(e) < 0$. This deals with (2) above.
- Therefore, SFM is as "easy" as moving from matroid rank functions to not-necessarily-integral polymatroidal functions.

Testing membership in polymatroids

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- And this is true iff $\min f(A) - x(A) \geq 0$.
- So, given a strongly polynomial time algorithm for general submodular function, we can test polyhedral membership, in at least this limited (polymatroidal polytope) sense.

Polymatroidal polyhedron (or a "polymatroid")

Recall from Lecture 7:

Definition 4.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For any $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (called a P-basis of x), has the same component sum $y(E)$. That is for any two maximal vectors $y^1, y^2 \in P$, we have $y^1(E) = y^2(E)$.

- A polymatroid is a compact set that is zero containing, down monotone, and any maximal vector y in P, bounded by another vector x, has the same vector rank.
- A matroid a set system that is empty-set containing, down closed, and any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank.

Polymatroidal polyhedron and greedy

- Recall greedy algorithm (from Lec 5): Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

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- For a matroid, we saw (Lec5) that set system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

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- Stated succinctly, considering $\max w(I) : I \in \mathcal{I}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.

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- That is, if we consider $\max wx : x \in P_f$, where P_f represent the "independent vectors", is it the case that P_f is a polymatroid iff greedy works for this maximization?

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- That is, if we consider $\max wx : x \in P_f$, where P_f represent the "independent vectors", is it the case that P_f is a polymatroid iff greedy works for this maximization?
- Can we even relax things so that $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting?

Polymatroidal polyhedron and greedy

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- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.

$$w(e_1) \geq w(e_2) \geq \dots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \dots \geq w(e_m)$$

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- Let $k + 1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1)}$.
- Next define partial accumulated sets E_i so that for $i = 0 \ldots m$, we have w.r.t. the above sorted order:

$$U_i \stackrel{\text{def}}{=} \{e_1, e_2, \ldots e_i\} \tag{9}$$

(note $U_0 = \emptyset$ and $f(U_0) = 0$, and U_i is always w.r.t w).

U_i^w

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- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(U_1) \tag{10}$$

$$x(e_i) \stackrel{\text{def}}{=} f(U_i) - f(U_{i-1}) \text{ for } i = 2 \dots k \tag{11}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \dots m = |E| \tag{12}$$

Polymatroidal polyhedron and greedy

Theorem 4.2

The vector $x \in \mathbb{R}_+^E$ as previously defined maximizes wx over P_f.

Proof.

. . .

Polymatroidal polyhedron and greedy

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Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f) = \min\left(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\right) \quad (13)$$

$y_A \in \mathbb{R}^+$ is a single scalar value

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- Define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{U_i} \stackrel{\text{def}}{=} w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{14}$$

$$y_E \stackrel{\text{def}}{=} w(e_m), \text{ and} \tag{15}$$

$$y_A = 0 \text{ otherwise} \tag{16}$$

↖ sorted w.

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Polymatroidal polyhedron and greedy

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- We first see that $x \in P_f$ (that is $x(A) \leq f(A), \forall A$) by induction on $|A|$. Clearly it holds for $A = \emptyset$.

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$x(A \setminus e_\ell) = x(A) - x(e_\ell)$

- And therefore,

$$x(A) \le f(A \setminus \{e_\ell\}) + x(e_\ell) = f(A \setminus \{e_\ell\}) + f(U_\ell) - f(U_{\ell-1}) \le f(A) \tag{18}$$

for $\ell \le k$,

where the last inequality follows by submodularity of f (if $\ell \le k$) and by monotonicity of f (if $\ell > k$) where $x(e_\ell) = 0$.

$f(A) + f(U_{\ell-1}) \ge f(U_\ell) + f(A \setminus e_\ell) = f(A \cup U_{\ell-1}) + f(A \cap U_{\ell-1})$

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  where the last inequality follows by submodularity of f (if $\ell \leq k$) and by monotonicity of f (if $\ell > k$) where $x(e_\ell) = 0$.
- So, therefore, we have $x \in P_f$.

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Polymatroidal polyhedron and greedy

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Proof.

- Now y is also feasible for the dual constraints in Eq. 13 since:

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Polymatroidal polyhedron and greedy

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Polymatroidal polyhedron and greedy

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The vector $x \in \mathbb{R}_+^E$ *as previously defined maximizes* wx *over* P_f.

Proof.

- Now y is also feasible for the dual constraints in Eq. 13 since:
- clearly, $y \geq 0$;
- also, considering y component wise, for any i, we have that

$$\sum_{A:e_i \in A} y_A = \sum_{j \geq i} y_{U_j} = w(e_i). \tag{19}$$

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- Now optimality for x and y follows from

$$wx = \sum_{e \in E} w(e)x(e) = \sum_{i=1}^{m} w(e_i)(f(U_i) - f(U_{i-1})) \tag{20}$$

$$= \sum_{i=1}^{n-1} f(U_i)(w(e_i) - w(e_{i+1})) + f(E)w(e_m) = \sum_{A \subseteq E} y_A f(A)$$

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Polymatroidal polyhedron and greedy

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Proof.

- The third equality (in Eq. 20) follows since

$$xw = \sum_{i=1}^{m} x_i w_i = \sum_{i=1}^{m} x_i \Big(\sum_{j=1}^{i} w(e_j) - \sum_{j=1}^{i-1} w(e_j) \Big) \quad (22)$$

$$= \sum_{i=1}^{m} x_i \Big(w(U_i) - w(U_{i-1}) \Big) \quad (23)$$

$$= \sum_{i=1}^{m} x_i w(U_i) - \sum_{i=1}^{m-1} x_{i+1} w(U_i) \quad (24)$$

$$= x_m w(U_m) + \sum_{i=1}^{m-1} (x_i - x_{i+1}) w(U_i) \quad (25)$$

□

Fix switch to correspond to prev slide.

Polymatroidal polyhedron and greedy

Theorem 4.3

Conversely, suppose P is a polytope of form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to $\max(wx : x \in P)$ is optimum only if f is submodular.

Proof.

- Name elements of E in arbitrary order $(e_1, e_2, \ldots, e_m)$ and define $E_i = (e_1, e_2, \ldots, e_i)$.

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- Define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\}$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\}$ for some $1 \leq p \leq q \leq m$.

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Polymatroidal polyhedron and greedy

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- Note, then $A \cap B = \{e_1, \ldots, e_k\}$.

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- Note, then $A \cap B = \{e_1, \dots, e_k\}$.
- Define w as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{26}$$

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Polymatroidal polyhedron and greedy

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- Suppose optimum solution x is given by the greedy procedure.

Polymatroidal polyhedron and greedy

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$$\sum_{i=1}^{k} x_i = f(U_1) + \sum_{i=2}^{k}(f(U_i) - f(U_{i-1})) = f(U_k) = f(A \cap B) \quad (27)$$

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Polymatroidal polyhedron and greedy

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-
$$\sum_{i=1}^{p} x_i = f(U_1) + \sum_{i=2}^{p} (f(U_i) - f(U_{i-1})) = f(U_p) f(A) \quad (28)$$

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Proof.

- Then
$$\sum_{i=1}^{k} x_i = f(U_1) + \sum_{i=2}^{k}(f(U_i) - f(U_{i-1})) = f(U_k) = f(A \cap B) \quad (27)$$
-
$$\sum_{i=1}^{p} x_i = f(U_1) + \sum_{i=2}^{p}(f(U_i) - f(U_{i-1})) = f(U_p) f(A) \quad (28)$$
-
$$\sum_{i=1}^{q} x_i = f(U_1) + \sum_{i=2}^{q}(f(U_i) - f(U_{i-1})) = f(U_q) = f(A \cup B) \quad (29)$$

...

Polymatroidal polyhedron and greedy

Theorem 4.3

Conversely, suppose P is a polytope of form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to $\max(wx : x \in P)$ is optimum only if f is submodular.

Proof.

- Thus, we have

$$\sum_{i:e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \tag{30}$$

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Polymatroidal polyhedron and greedy

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Polymatroidal polyhedron and greedy

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- Thus, we have

$$\sum_{i:e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \le f(B) \quad (30)$$

- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P)$, we have that $x \in P$.
- Thus,

$$x(B) = f(A \cup B) + f(A \cap B) - f(A) = \overbrace{\sum_{i:e_i \in B} x_i}^{x(B)} \le f(B) \quad (31)$$

ensuring the submodularity of f, since A and B are arbitrary. □

Polymatroidal polyhedron and greedy

- Thus, summarizing this into the complete theorem, we have a result very similar to matroids.

Theorem 4.4

If $f : 2^E \rightarrow \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

$\forall w$

An extension of f

- We may consider the optimization a function $\tilde{f} : \mathbb{R}^E \to \mathbb{R}$ as

$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{32}$$

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- Then, for any w, from the above theorem, we can compute this function using the greedy algorithm.
- That is, we have
$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{33}$$
$$= \sum_{i=1}^{m} w(e_i)(f(U_i) - f(U_{i-1})) \tag{34}$$
$$= w(e_m)f(U_m) + \sum_{i=1}^{m-1}(w(e_i) - w(e_{i+1}))f(U_i) \tag{35}$$

when f is submodular.

where $U_i = \{e_1, e_2, \ldots, e_i\}$ based on the elements of E being named, w.l.o.g., in order of decreasing w, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

An extension of f

- Moreover, from $\tilde{f}$ we can recover f.

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- Moreover, from $\tilde{f}$ we can recover f.
- Take $w = \mathbf{1}_A$ for some $A \subseteq E$.
- Then, we order w so that $1_A(i) = 1$ if $i \leq |A|$, and $1_A(i) = 0$ otherwise.
- This gives

$$\begin{aligned}
\tilde{f}(w) &= \max(\mathbf{1}_A x : x \in P_f) && (36)\\
&= \mathbf{1}_A(m) f(U_m) + \sum_{i=1}^{m-1} (\mathbf{1}_A(i) - \mathbf{1}_A(i+1)) f(U_i) && (37)\\
&= (\mathbf{1}_A(|A|) - \mathbf{1}_A(|A|+1)) f(U_i) && (38)\\
&= f(A) && (39)
\end{aligned}$$

An extension of f

$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{40}$$

- Therefore, if f is a submodular function, we can write

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(U_i) \tag{41}$$

where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted according to w as before.

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- Clearly, $\tilde{f}(w)$ is always convex in w, since it is the maximum of a set of linear functions. (true even when f is not submodular) when written as eq. (40).

An extension of f

- Recall, for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \underbrace{(w_1 - w_2)}_{\lambda_1} \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{v_1} + \underbrace{(w_2 - w_3)}_{\lambda_2} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{v_2} + \cdots + \underbrace{(w_{n-1} - w_n)}_{\lambda_{n-1}} \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}}_{v_{n-1}} + \underbrace{(w_n)}_{\lambda_n} \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}}_{v_n} \tag{42}$$

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- If we take w in decreasing order, then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

An extension of f

- Define sets U_i based on this decreasing order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots, e_i\} \tag{43}$$

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$$U_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots, e_i\} \tag{43}$$

- Note that

$$\mathbf{1}_{U_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{1}_{U_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{1}_{U_\ell} = \begin{pmatrix} \left.\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\} \ell\times \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} (n-\ell)\times \end{pmatrix}, \text{ etc.} \tag{44}$$

An extension of f

- Thus, for any f, we can define an extension in this way, with

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(U_i) \tag{45}$$

with the U_i's and sorted order of w defined as above, so that $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{U_i}$

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- Lovász showed that if a function $\tilde{f}(w)$, so defined is convex, then the underlying f must be submodular.
- This "extension" of f, in any case, is called the Lovász extension of f.

"Edmonds-Lovasz" extension ?

Polymatroidal polyhedron and greedy

Theorem 5.1

A function $f : 2^E \to \mathbb{R}$ is submodular iff its Lovász extension $\tilde{f}$ of f is convex.

Proof.

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- We note that, based on the extension definition, $\tilde{f}(\alpha w) = \alpha \tilde{f}(w)$ for any $\alpha \in \mathbb{R}_+$. I.e., f is a positively homogeneous convex function.
- Given $A, B \subseteq E$, we have that

$$\tilde{f}(\mathbf{1}_A + \mathbf{1}_B) = \tilde{f}(\mathbf{1}_{A \cup B} + \mathbf{1}_{A \cap B}) \tag{46}$$

$$= f(A \cup B) + f(A \cap B). \tag{47}$$

Exercise: show this.

...

Polymatroidal polyhedron and greedy

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A function $f : 2^E \to \mathbb{R}$ is submodular iff its Lovász extension $\tilde{f}$ of f is convex.

Proof.

- Also, since $\tilde{f}$ is convex, we have

$$\tilde{f}(0.5\mathbf{1}_A + 0.5\mathbf{1}_B) \leq 0.5\tilde{f}(\mathbf{1}_A) + 0.5\tilde{f}(\mathbf{1}_B) \quad (48)$$

$$= 0.5(f(A) + f(B)) \quad (49)$$

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- Thus,

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \tag{50}$$

as required..

□

Scratch Paper

1. What did Edmonds, [illegible]/Wolfe, Lovász show
Add slide.

Scratch Paper

Scratch Paper

Sources for Today's Lecture

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