EE595A – Submodular functions, their optimization and applications – Spring 2011

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering

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http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/

Lecture 10 - May 4th, 2011

Announcements

- On Final projects. **One** single page final project proposals (revision one) are due next Friday (one week from today) at 6:00pm.
- Again, all submissions must be done electronically, via our drop box. See the link `https://catalyst.uw.edu/collectit/dropbox/bilmes/14888`, or look at the homework on the web page.
- Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.
- Ideal proposal would, say, lead to a NIPS paper in June and be related to submodularity.

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- This corresponds to $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min \{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min \{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).

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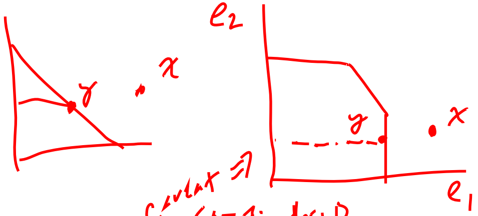
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*



convex ⇒

Careful: this is polymatroid but today we are talking about $P_{\{\text{ind set}\}}$

$y(A) = r_M(A)$

$y(E \setminus A) = x(E \setminus A)$

$A = \{e_1\}$

$E \setminus A = \{e_2\}$

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- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).

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- This will also run in polynomial time.

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- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It has taken us a few lectures to fully develop this algorithm, today we will probably finish it.

Matroid Partition Problem

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 2.1 Edmonds 1964

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{2}$$

where r_i is the rank function of M_i.

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$$|A| \leq kr(A) \;\; \forall A \subseteq E \tag{3}$$
- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as
$$\frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{4}$$

Matroid Partition Problem and Submodular Function Minimization

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- We also see that this is essentially a special case of submodular function minimization, namely finding A that minimizes $r(A) - \frac{1}{k}\mathbf{1}(A)$.
- In the general case, we are looking for an A that minimizes $\sum_i r_i(A) - \mathbf{1}(A)$, and a sum of submodular functions is submodular (in fact, a sum of matroid rank functions is a type of polymatroid rank function Exercise).

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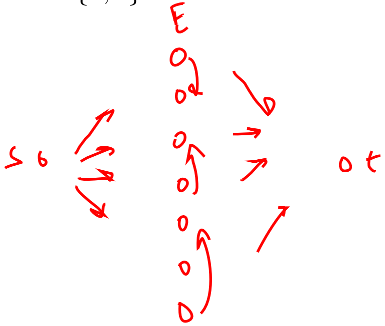

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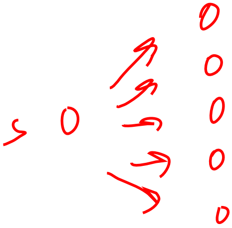

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- Create directed edge (e, t) for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ **and** $I_i + e \in \mathcal{I}$. I.e., we add this edge (e, t) if there is some independent set I_i that remains independent if e is added to it.

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- Add directed edge (e, f) for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some i. That is, we add an edge (e, f) where e directs **to** the elements of a (nec. unique) circuit that is potentially created when e is added to I_i for some i.

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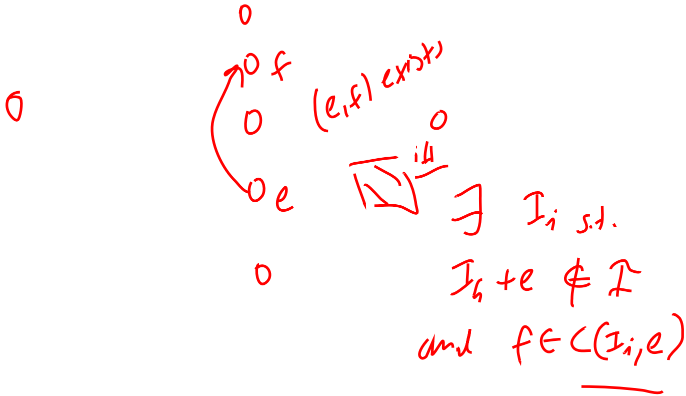
- Therefore, incoming edges to e are either from source s, or from some other node that created a circuit in some I_i.



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so $S_j + e \in \mathcal{I}$, $i \neq j$

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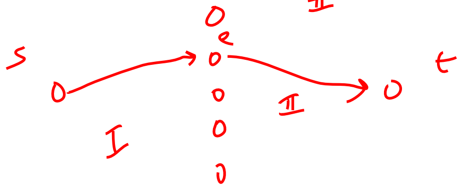

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- If the shortest path is $S = (s, e, t)$ then we can add e to some independent set and it is still independent.
- If the shortest path is $S = (s, e, f, t)$ then we can add e to some I_1, create a circuit, but that gets broken when we remove f from that circuit rendering I_1 once again independent, but then there must be some other I_2 that f can be added to w/o making I_2 independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress since overall, e is added.

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  4. subtract f_2 from I_2, eliminating the circuit C_2.
  5. add f_2 to some I_3, not making a circuit due to edge (f_2, t).

  thus making progress.

Matroid Partition - Flow solution when $M = M_i, \forall i$

- Note that $I_1 \neq I_2$ since the edge (f, t) meant that we originally had $f \notin I_2$.
- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
  1. add e to some I_1, thus making a circuit C_1 due to edge (e, f_1).
  2. subtract f_1 from I_1, eliminating the circuit C_1.
  3. add f_1 to some I_2, thus making a circuit C_2 due to edge (f_1, f_2).
  4. subtract f_2 from I_2, eliminating the circuit C_2.
  5. add f_2 to some I_3, not making a circuit due to edge (f_2, t).

  thus making progress.
- Here, $I_1 \neq I_2$, and $I_2 \neq I_3$, but could have $I_1 = I_3$ Exercise:

Flow solution theorem

Thus, we have outlined the proof of one direction in the following theorem. When all matroids are the same $\forall i, M_i = M$ for some matroid, we have:

Theorem 3.1

There is an (s, t) path in the aforementioned graph iff the set of independent sets $(I_i : i \in J)$ can be grown by one element and still be a partition of some subset of E.

The other direction can be shown as a consequence of Theorem 2.1. Exercise

Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*
- By the above theorem, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).

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- We gradually build up y by adding new independent sets (and augmenting J), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.

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- The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$.

Augmenting path theorem

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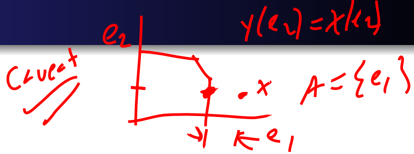

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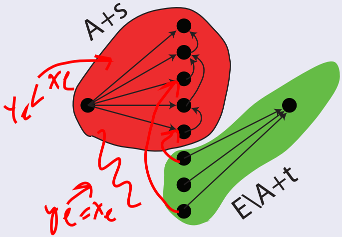

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9. we have that $I_i + e \in \mathcal{I}$, implying that (e, t) is an edge in G (impossible since $(s, e) \in G$, so can't also have $(e, t) \in G$ since no s, t path in G).

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10. alternatively, $I_i + e \notin \mathcal{I}$, so circuit $C(I_i, e)$ exists which can not be contained in A. *(we needed in (7) that $(I_i \cap A) + e$ is independent, and if the circuit was fully in A then this independence consequence would not hold).*

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12. Therefore, since $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$, we have:

$$y(A) = \sum_{a \in A} y_a = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}(A) \tag{6}$$

$$= \sum_{i \in J} \lambda_i |I_i \cap A| = \sum_{i \in J} \lambda_i r(A) = r(A) \qquad \text{as required.} \tag{7}$$

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Proof of Thm 4.1.

13. Conversely, suppose there is a directed s, t path in G, and it is given by sequence $S = (e_1, e_2, \ldots, e_m)$ of distinct elements. $e_i \in E$

$S = s = e_0, e_1, \ldots, e_m, e_{m+1} = t$

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$\subset \{1, \ldots, m\}$

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$$(e_1, e_2) \in G \Rightarrow e_2 \in C(I_{i(1)}, e_1); \tag{9}$$

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$$(e_j, e_{j+1}) \in G \Rightarrow e_{j+1} \in C(I_{i(j)}, e_j) \text{ for } 1 \leq j \leq m-1; \tag{10}$$

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Proof of Thm 4.1.

13. Conversely, suppose there is a directed s, t path in G, and it is given by ordered sequence $S = (e_1, e_2, \dots, e_m)$ of distinct elements.
14. Then there is a mapping $i : [m] \to J$ (where the $i(j)$ are not distinct), satisfying — distinct w.r.t. S

$$(s, e_1) \in G \Rightarrow x_{e_1} > y_{e_1}; \tag{8}$$

$$(e_1, e_2) \in G \Rightarrow e_2 \in C(I_{i(1)}, e_1); \tag{9}$$

$$(e_j, e_{j+1}) \in G \Rightarrow e_{j+1} \in C(I_{i(j)}, e_j) \text{ for } 1 \leq j \leq m-1; \tag{10}$$

$$(e_m, t) \in G \Rightarrow I_{i(m)} + e_m \in \mathcal{I}. \tag{11}$$

Augmenting path theorem

Theorem 4.1

If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

Proof of Thm 4.1.

$i_s(j)$

15. Now, for $i \in J$, define $k_i = |\{j : i = i(j)\}| = \cup_{j \in J} 1_{i=i(j)}$ be the number of times that the i'th independent set I_i is used in the mapping $i : [m] \to J$.

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18. Next, we add e_1 to $\cup_i I_i$, and distribute amongst them to remove any circuits, as follows.

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for $j \in 1 \dots m-1$ **do**
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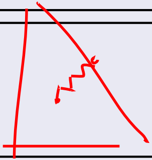

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21. The theorem is proven.

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Augmenting path theorem consequences

Corollary 4.2

For any $x \in \mathbb{R}_+^E$, *we have*

$$\max\left(y(E) : y \le x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (12)$$

Note: this was not used in the theorem above, rather it is a consequence!

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5. Then, there is a set A such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, or that $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality.

□

Augmenting path theorem consequences

Corollary 4.3

Given matroid M, we have

$$P_{ind.\ set} = P_r \tag{14}$$

We even get this a consequence!

Proof.

- We saw before (in lecture 7) that this follows from corollary 4.2 (which we encountered in lecture 7).

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- Therefore, the equivalence follows indirectly just from Theorem 4.1!!

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- Key in this is to: 1) scan nodes in the order that they are labeled, and 2) label nodes (from a node being scanned) in an order consistent with some fixed total order on all vertices.
- While 1) ensures that the path has as few edges as possible (proven in Edmonds/Karp), 2) results in a lexicographically minimum order. Both together are called a *consistent breadth-first search*, or CBFS.

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- On our current context, we have results quite similar to this that guarantee that the number of augmentations is polynomially bounded, yielding our next theorem.

Bounding the number of augmenting paths

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Let $G_0, G_1, \dots, G_k$ be a sequence of digraphs, each having vertex set $E \cup \{s, t\}$, and correspond to such graphs each one running the algorithm implied by theorem 4.1 Assume fixed total order of $E \cup \{s\}$. Let Q_i denote the CBFS path in G_i, for $0 \leq i < k$. If it is the case that, for $0 \leq i < k$:

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Then we have that the number of augmentations has bound $k \leq |E|^3$.

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It is possible to construct an augmentation scheme such that each augmenting path is done in accordance to Theorem 4.4. Each such augmentation is CBFS, and is called a "grand" augmentation, and is maximal in a certain way. This achieves the $O(n^3)$ time, in the number of augmentations, mentioned above.

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- On the other hand, this algorithm has some intriguing properties.

Towards SFM

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  3. This works only for the difference between r and x, but we'd like an algorithm that works for any arbitrary submodular function f, even non-monotone and/or non-non-increasing/decreasing.
- It turns out that (2) and (3) is easy to deal with, but (1) took another 16 years to solve (and perhaps can still be seen as unsolved, w.r.t. wanting a scalable algorithm).

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- Also, f is monotone non-decreasing and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since

$$f(B + v) - f(B) = g(B + v) - g(B) + m(v) \tag{16}$$

$$= g(B + v) - g(B) + g(E - v) - g(E) \tag{17}$$

$$\geq 0 \tag{18}$$

since, by submodularity, $g(B + v) - g(B) \geq g(E - v) - g(E)$.

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- This follows since f is monotone non-decreasing, and $m(A) = m(A') + m(e)$. This deals with (2) above.
- Therefore, SFM is as "easy" as moving from matroid rank functions to not-necessarily-integral polymatroidal functions.

Scratch Paper

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Sources for Today's Lecture

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