On Final projects. **One** single page final project proposals (revision one) are due next Friday (one week from today) at 6:00pm.

Again, all submissions must be done electronically, via our drop box. See the link
https://catalyst.uw.edu/collectit/dropbox/bilmes/14888, or look at the homework on the web page.

Email me and/or stop by office hours for ideas. The proposals next Friday are non-binding (you can change your mind later) but you should start thinking about project proposals now.

Ideal proposal would, say, lead to a NIPS paper in June and be related to submodularity.
Most violated inequality problem

Consider

$$P_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\}$$  \hspace{1cm} (1)
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- Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \not\in P_r \).

- The most violated inequality when \( x \) is considered w.r.t. \( P_r \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e.,
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This corresponds to \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \).
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More importantly, \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) a form of submodular function minimization, namely

\[ \min \{ r_M(A) - x(A) : A \subseteq E \} \] for a submodular function consisting of a difference of matroid rank and modular (so no longer nec. monotone, nor positive).
Problem To Solve

In particular, we will solve the following problem:

- Given a matroid \( M = (E, \mathcal{I}) \) along with an independence testing oracle (i.e., for any \( A \subseteq E \), tells us if \( A \in \mathcal{I} \) or not), and a vector \( x \in \mathbb{R}_+^E \);
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- find: a maximizing $y \in \mathcal{P}_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. Of course, for any such $y$ we must have that $y(E) \leq r(A) + x(E \setminus A)$.

![Diagram](image_url)
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By the above theorem, the existence of such an $A$ will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, $A$ is minimal in terms of $f(A) \overset{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
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- This will also run in polynomial time.
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$. 


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- It has taken us a few lectures to fully develop this algorithm, today we will probably finish it.
Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$. 
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We wish to, if possible, partition $E$ into $k$ blocks, $l_i, i \in \{1, 2, \ldots, k\}$ where $l_i \in \mathcal{I}_i$. 
Matroid Partition Problem

- Suppose $M_i = (E, I_i)$ is a matroid and that we have $k$ of them on the same ground set $E$.
- We wish to, if possible, partition $E$ into $k$ blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in I_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.
Theorem 2.1  

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$

(2)

where $r_i$ is the rank function of $M_i$. 
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Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

$$|A| \leq kr(A) \ \forall A \subseteq E$$  \hspace{1cm} (3)
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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k} \mathbf{1}(A) \leq r(A) \ \forall A \subseteq E$$

(4)
Matroid Partition Problem and Submodular Function Minimization

- Recall definition of matroid polytope

\[ P_r = \left\{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \]  

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- Then we see that this special case of the matroid partition problem is just testing if \( \frac{1}{k} \mathbf{1} \in P_r \), a problem of testing the membership in matroid polyhedra.
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- We also see that this is essentially a special case of submodular function minimization, namely finding \( A \) that minimizes \( r(A) - \frac{1}{k} \mathbf{1}(A) \).
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In the general case, we are looking for an \( A \) that minimizes \( \sum_i r_i(A) - \mathbf{1}(A) \), and a sum of submodular functions is submodular (in fact, a sum of matroid rank functions is a type of polymatroid rank function Exercise).
Matroid Partition - Flow solution when $M = M_i, \forall i$

- It extends partition $(I_i : i \in J)$ of a proper subset of $E$ into $k$ independent sets, to such a partitioning of a larger subset.
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- It extends partition $(I_i : i \in J)$ of a proper subset of $E$ into $k$ independent sets, to such a partitioning of a larger subset.
- At each step, we construct an auxiliary digraph graph $G$ for this problem.
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- It extends partition $(I_i : i \in J)$ of a proper subset of $E$ into $k$ independent sets, to such a partitioning of a larger subset.
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- Vertex set is $E \cup \{s, t\}$ where $s$ source and $t$ sink are new nodes.

![Digraph Diagram]
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- Vertex set is $E \cup \{s, t\}$ where $s$ source and $t$ sink are new nodes.
- Create directed edge $(s, e)$ for all $e \in E$ such that $e \notin \cup_{i \in J} I_i$. That is, any element not yet in one of the independent sets.
- Create directed edge $(e, t)$ for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ and $I_i + e \in C(I_i, e)$. I.e., we add this edge $(e, t)$ if there is some independent set $I_i$ that remains independent if $e$ is added to it.
- Add directed edge $(e, f)$ for any distinct $e, f \in E$ such that $I_i + e \notin I_i$ and $f \in C(I_i, e)$ for some $i$. That is, we add an edge $(e, f)$ where $e$ directs to the elements of a (nec. unique) circuit that is potentially created when $e$ is added to $I_i$ for some $i$. 
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Therefore, incoming edges to $e$ are either from source $s$, or from some other node that created a circuit in some $I_i$. If the shortest path is $S = (s, e, t)$ then we can add $e$ to some independent set and it is still independent. If the shortest path is $S = (s, e, f, t)$ then we can add $e$ to some $I_1$, create a circuit, but that gets broken when we remove $f$ from that circuit rendering $I_1$ once again independent, but then there must be some other $I_2$ that $f$ can be added to w/o making $I_2$ independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress since overall, $e$ is added.
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- Therefore, incoming edges to $e$ are either from source $s$, or from some other node that created a circuit in some $I_i$.
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- So the outgoing edges from $e$ either: 1) correspond to an independent set $e$ may be added to, or 2) are to the circuit elements created when $e$ is added to an independent set.
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- If the shortest path is $S = (s, e, f, t)$ then we can add $e$ to some $I_1$, create a circuit, but that gets broken when we remove $f$ from that circuit rendering $I_1$ once again independent, but then there must be some other $I_2$ that $f$ can be added to w/o making $I_2$ independent. Thus, the new independent sets are $I_1 + e - f$ and $I_2 + f$, thus we are making progress since overall, $e$ is added.
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Matroid Partition - Flow solution when $M = M_i, \forall i$

- Note that $I_1 \neq I_2$ since the edge $(f, t)$ meant that we originally had $f \notin I_2$.
- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
  1. add $e$ to some $I_1$, thus making a circuit $C_1$ due to edge $(e, f_1)$.
  2. subtract $f_1$ from $I_1$, eliminating the circuit $C_1$.
  3. add $f_1$ to some $I_2$, thus making a circuit $C_2$ due to edge $(f_1, f_2)$.
  4. subtract $f_2$ from $I_2$, eliminating the circuit $C_2$.
  5. add $f_2$ to some $I_3$, not making a circuit due to edge $(f_2, t)$.

thus making progress.

Here, $I_1 \neq I_2$, and $I_2 \neq I_3$, but could have $I_1 = I_3$.

Exercise:
Matroid Partition - Flow solution when $M = M_i, \forall i$

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  3. add $f_1$ to some $I_2$, thus making a circuit $C_2$ due to edge $(f_1, f_2)$.
  4. subtract $f_2$ from $I_2$, eliminating the circuit $C_2$.

...
Matroid Partition - Flow solution when $M = M_i, \forall i$

- Note that $I_1 \neq I_2$ since the edge $(f, t)$ meant that we originally had $f \not\in I_2$.

- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
  1. add $e$ to some $I_1$, thus making a circuit $C_1$ due to edge $(e, f_1)$.
  2. subtract $f_1$ from $I_1$, eliminating the circuit $C_1$.
  3. add $f_1$ to some $I_2$, thus making a circuit $C_2$ due to edge $(f_1, f_2)$.
  4. subtract $f_2$ from $I_2$, eliminating the circuit $C_2$.
  5. add $f_2$ to some $I_3$, not making a circuit due to edge $(f_2, t)$.

thus making progress.
Matroid Partition - Flow solution when $M = M_i, \forall i$

- Note that $l_1 \neq l_2$ since the edge $(f, t)$ meant that we originally had $f \notin l_2$.
- If the shortest path is $S = (s, e, f_1, f_2, t)$ then we can:
  1. add $e$ to some $l_1$, thus making a circuit $C_1$ due to edge $(e, f_1)$.
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  3. add $f_1$ to some $l_2$, thus making a circuit $C_2$ due to edge $(f_1, f_2)$.
  4. subtract $f_2$ from $l_2$, eliminating the circuit $C_2$.
  5. add $f_2$ to some $l_3$, not making a circuit due to edge $(f_2, t)$.
thus making progress.
- Here, $l_1 \neq l_2$, and $l_2 \neq l_3$, but could have $l_1 = l_3$ Exercise:
Thus, we have outlined the proof of one direction in the following theorem. When all matroids are the same $\forall i, M_i = M$ for some matroid, we have:

**Theorem 3.1**

There is an $(s, t)$ path in the aforementioned graph iff the set of independent sets $(I_i : i \in J)$ can be grown by one element and still be a partition of some subset of $E$.

The other direction can be shown as a consequence of Theorem 2.1.

**Exercise**
Problem To Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}^E_+$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. Of course, for any such $y$ we must have that $y(E) \leq r(A) + x(E \setminus A)$.
- By the above theorem, the existence of such an $A$ will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, $A$ is minimal in terms of $f(A) \overset{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of $f$ at $A$.
- This will also run in polynomial time.
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$. 
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$.

- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{I_i}$. 
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$.
- We keep a family of independent sets $(l_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{l_i}$.
- Therefore, $y \in P_{\text{ind. set}}$.
- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$.

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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.

- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
Idea of the algorithm

- Given $x \in \mathbb{R}_+$, we build up $y$ from the ground up, ensuring that $y \leq x$, and starting with $y = 0$.

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- Therefore, $y \in P_{\text{ind. set}}$.

- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.

- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).

- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we’ll keep $y \leq x$. 
Define associated digraph $G$ as follows.

- Create a directed edge $(s, e)$ for all $e \in E$ such that $y(e) < x(e)$.
- Create a directed edge $(e, t)$ for all $e \in E$ such that there exists $i \in J$ with $e \notin I_i$ and $I_i + e \in I$.
- Add a directed edge $(e, f)$ for any distinct $e, f \in E$ such that $I_i + e \in I$ and $f \in C(I_i, e)$ for some $i$. That is, we add an edge $(e, f)$ where $e$ directs to the elements of a (nec. unique) circuit that is potentially created when $e$ is added to $I_i$ for some $i$. 

The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$. 

Associated digraph for polyhedra membership
Define associated digraph \( G \) as follows.

- Vertices of \( G \), \( V(G) = E \cup \{s, t\} \) where \( s, t \) are distinct elements not in \( E \).
Define associated digraph $G$ as follows.

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- Vertices of $G$, $V(\ G\ ) = E \cup \{s, t\}$ where $s, t$ are distinct elements not in $E$.
- Create a directed edge $(s, e)$ for all $e \in E$ such that $y(e) < x(e)$. Intuitively, $y$ is our current measure of an “independence” of sorts, and any $e$ s.t. $y(e) < x(e)$ is not yet “saturated”
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Associated digraph for polyhedra membership

- Define associated digraph $G$ as follows.
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  - Create directed edge $(e, t)$ for all $e \in E$ such that $\exists i \in J$ with $e \notin l_i$ and $l_i + e \in \mathcal{I}$. I.e., we add this edge $(e, t)$ if there is some independent set $l_i$ that remains independent if $e$ is added to it.
Define associated digraph $G$ as follows.

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Create a directed edge $(s, e)$ for all $e \in E$ such that $y(e) < x(e)$. Intuitively, $y$ is our current measure of an “independence” of sorts, and any $e$ s.t. $y(e) < x(e)$ is not yet “saturated” (compare with: any $e \notin \bigcup_i I_i$ from matroid partition case).

Create directed edge $(e, t)$ for all $e \in E$ such that $\exists i \in J$ with $e \notin I_i$ and $I_i + e \in \mathcal{I}$. I.e., we add this edge $(e, t)$ if there is some independent set $I_i$ that remains independent if $e$ is added to it.

Add directed edge $(e, f)$ for any distinct $e, f \in E$ such that $I_i + e \notin \mathcal{I}$ and $f \in C(I_i, e)$ for some $i$. That is, we add an edge $(e, f)$ where $e$ directs to the elements of a (nec. unique) circuit that is potentially created when $e$ is added to $I_i$ for some $i$. 
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Add directed edge $(e, f)$ for any distinct $e, f \in E$ such that $l_i + e \notin \mathcal{I}$ and $f \in C(l_i, e)$ for some $i$. That is, we add an edge $(e, f)$ where $e$ directs to the elements of a (nec. unique) circuit that is potentially created when $e$ is added to $l_i$ for some $i$.

The algorithm starts with $y = 0$, $J = \{0\}$, $I_0 = \emptyset$, and $\lambda_0 = 1$. 
Thus, we consider $x \in \mathbb{R}_+$. 

Theorem 4.1

If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$. 

Augmenting path theorem
Augmenting path theorem

- Thus, we consider $x \in \mathbb{R}_+$.  

- We’ve constructed the aforementioned $s, t$ graph $G$ as previously mentioned, where for each $e \in E$, we’ve got a node in $G$, along with additional nodes (and edges) $s, t$. 

Augmenting path theorem

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- We maintain $y = \sum_{i \in J} \lambda_i \mathbf{1}_{\mathbf{I_i}} \leq x$ and thus $y \in P_{\text{ind. set}}$. 

Augmenting path theorem

- Thus, we consider $x \in \mathbb{R}_+$.
- We’ve constructed the aforementioned $s, t$ graph $G$ as previously mentioned, where for each $e \in E$, we’ve got a node in $G$, along with additional nodes (and edges) $s, t$.
- We maintain $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i} \leq x$ and thus $y \in P_{\text{ind. set}}$.
- From this, we can obtain the following theorem (most violated inequality, then, is $e \in E$ s.t. $x(e) > y(e)$).
Augmenting path theorem

Thus, we consider $x \in \mathbb{R}_+$. We’ve constructed the aforementioned $s, t$ graph $G$ as previously mentioned, where for each $e \in E$, we’ve got a node in $G$, along with additional nodes (and edges) $s, t$. We maintain $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i} \leq x$ and thus $y \in P_{\text{ind. set}}$. From this, we can obtain the following theorem (most violated inequality, then, is $e \in E$ s.t. $x(e) > y(e)$).

Theorem 4.1

If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$. 
Augmenting path theorem

Theorem 4.1

If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).

Proof of Thm 4.1.

1. First, assume that there is no such s, t path in G.
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists y′ ∈ P with y ≤ y′ ≤ x, with y′(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

**Proof of Thm 4.1.**

1. First, assume that there is no such s, t path in G.
2. Choose the set A ⊆ E such that no edge of G leaves the set {s} ∪ A. That is, choose A minimally corresponding to the zero-edge cut (A + s, E \ A + t).
Augmenting path theorem

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If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

**Proof of Thm 4.1.**

1. First, assume that there is no such $s, t$ path in $G$.
2. Choose the set $A \subseteq E$ such that no edge of $G$ leaves the set $\{s\} \cup A$. That is, choose $A$ minimally corresponding to the zero-edge cut $(A + s, E \setminus A + t)$.
3. Consider $E \setminus A$, and consider any $e \in E \setminus A$, then we must have $y_e = x_e$, or otherwise (if $y_e < x_e$) we'd have an edge $(s, e)$ leaving $s$. 

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Prof. Jeff Bilmes
Augmenting path theorem

Theorem 4.1

*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

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2. Choose the set A ⊆ E such that no edge of G leaves the set {s} ∪ A. That is, choose A minimally corresponding to the zero-edge cut (A + s, E \ A + t).
3. Consider E \ A, and consider any e ∈ E \ A, then we must have ye = xe, or otherwise (if ye < xe) we’d have an edge (s, e) leaving s.
4. Therefore, y(E \ A) = x(E \ A).
Augmenting path theorem

**Theorem 4.1**

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**Proof of Thm 4.1.**

5 Now, consider A. Thus, for all e ∈ A, there is an edge (s, e) and ye < xe (by graph construction).
Augmenting path theorem

Theorem 4.1

If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).

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5 Now, consider \( A \). Thus, for all \( e \in A \), there is an edge \((s, e)\) and \( y_e < x_e \) (by graph construction).

6 For all \( i \in J \), we claim that \(|I_i \cap A| = r(A)\) (which means that for all \( i \in J \) and all \( e \in A \setminus I_i \), \((I_i \cap A) + e \notin I\)).
Augmenting path theorem

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*If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).*

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7. If (6) is not true, then \( \exists i \in J \) and \( e \in A \setminus I_i \) s.t. \((I_i \cap A) + e \in \mathcal{I})\).
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

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1. Now, consider A. Thus, for all e ∈ A, there is an edge (s, e) and ye < xe (by graph construction).

2. For all i ∈ J, we claim that |I_i ∩ A| = r(A) (which means that for all i ∈ J and all e ∈ A \ I_i, (I_i ∩ A) + e ∉ I).

3. If (6) is not true, then ∃i ∈ J and e ∈ A \ I_i s.t. (I_i ∩ A) + e ∈ I.

4. There are two way for (7) to happen ((9) or (10)). That is, either:
Augmenting path theorem

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*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

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5. Now, consider A. Thus, for all e ∈ A, there is an edge (s, e) and y_e < x_e (by graph construction).

6. For all i ∈ J, we claim that |l_i ∩ A| = r(A) (which means that for all i ∈ J and all e ∈ A \ l_i, (l_i ∩ A) + e ∉ I).

7. If (6) is not true, then ∃i ∈ J and e ∈ A \ l_i s.t. (l_i ∩ A) + e ∈ I.

8. There are two way for (7) to happen ((9) or (10)). That is, either:

9. we have that l_i + e ∈ I, implying that (e, t) is an edge in G (impossible since (s, e) ∈ G, so can’t also have (e, t) ∈ G since no s, t path in G).
Augmenting path theorem

Theorem 4.1

If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

Proof of Thm 4.1.

Alternatively, $l_i + e \notin I$, so circuit $C(l_i, e)$ exists which can not be contained in $A$. (we needed in (7) that $(l_i \cap A) + e$ is independent, and if the circuit was fully in $A$ then this independence consequence would not hold).
Augmenting path theorem

Theorem 4.1

If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).

Proof of Thm 4.1.

10 Alternatively, \( l_i + e \notin \mathcal{I} \), so circuit \( C(l_i, e) \) exists which can not be contained in \( A \).

11 Thus, in this case, there exists \( f \in l_i \setminus A \) such that \( f \in C(l_i, e) \), and also edge \((e, f) \in G \). But this also can’t happen since this would be a zero-edge cut crossing edge. Thus, (6) is true.
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).*

**Proof of Thm 4.1.**

10. alternatively, \( l_i + e \notin \mathcal{I} \), so circuit \( C(l_i, e) \) exists which can not be contained in \( A \).

11. Thus, in this case, there exists \( f \in l_i \setminus A \) such that \( f \in C(l_i, e) \), and also edge \((e, f) \in G\). But this also can't happen since this would be a zero-edge cut crossing edge. Thus, (6) is true.

12. Therefore, since \( y = \sum_{i \in J} \lambda_i 1_{l_i} \), we have:

\[
y(A) = \sum_{a \in A} y_a = \sum_{i \in J} \lambda_i 1_{l_i}(A) \tag{6}
\]

\[
= \sum_{i \in J} \lambda_i |l_i \cap A| = \sum_{i \in J} \lambda_i r(A) = r(A) \quad \text{as required.} \tag{7}
\]
Augmenting path theorem

**Theorem 4.1**

If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

**Proof of Thm 4.1.**

Conversely, suppose there is a directed $s, t$ path in $G$, and it is given by sequence $S = (e_1, e_2, \ldots, e_m)$ of distinct elements. $e_i \in E$

$$S = s = e_0 \prec e_1, \ldots \prec e_m, e_{m+1} = t$$
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).*

**Proof of Thm 4.1.**

13 Conversely, suppose there is a directed \( s, t \) path in \( G \), and it is given by sequence \( S = (e_1, e_2, \ldots, e_m) \) of distinct elements.

14 Then there is a mapping \( i : [m] \to J \) (where the \( i(j) \) are not distinct), satisfying

\[
\text{31, \ldots, m} \]
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

**Proof of Thm 4.1.**

**13** Conversely, suppose there is a directed s, t path in G, and it is given by sequence $S = (e_1, e_2, \ldots, e_m)$ of distinct elements.

**14** Then there is a mapping $i : [m] \rightarrow J$ (where the $i(j)$ are not distinct), satisfying

$$(s, e_1) \in G \Rightarrow x_{e_1} > y_{e_1};$$

(8)
Augmenting path theorem

**Theorem 4.1**

If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

**Proof of Thm 4.1.**

13 Conversely, suppose there is a directed $s, t$ path in $G$, and it is given by sequence $S = (e_1, e_2, \ldots, e_m)$ of distinct elements.

14 Then there is a mapping $i : [m] \rightarrow J$ (where the $i(j)$ are not distinct), satisfying

\[
(s, e_1) \in G \Rightarrow x_{e_1} > y_{e_1}; \quad (8)
\]

\[
(e_1, e_2) \in G \Rightarrow e_2 \in C(i_{(1)}, e_1); \quad (9)
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Augmenting path theorem

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*If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.***

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\begin{align*}
(s, e_1) \in G &\implies x_{e_1} > y_{e_1}; & (8) \\
(e_1, e_2) \in G &\implies e_2 \in C(l_{i(1)}, e_1); & (9) \\
(e_j, e_{j+1}) \in G &\implies e_{j+1} \in C(l_{i(j)}, e_j) \text{ for } 1 \leq j \leq m - 1; & (10)
\end{align*}
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

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- $(s, e_1) \in G \Rightarrow x_{e_1} > y_{e_1}$; \hspace{1cm} (8)
- $(e_1, e_2) \in G \Rightarrow e_2 \in C(l_{i(1)}, e_1)$; \hspace{1cm} (9)
- $(e_j, e_{j+1}) \in G \Rightarrow e_{j+1} \in C(l_{i(j)}, e_j)$ for $1 \leq j \leq m - 1$; \hspace{1cm} (10)
- $(e_m, t) \in G \Rightarrow l_{i(m)} + e_m \in \mathcal{I}$. \hspace{1cm} (11)
Augmenting path theorem

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*If there is a directed path from s to t in G, then there exists y' ∈ P with y ≤ y' ≤ x, with y'(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

**Proof of Thm 4.1.**

Now, for i ∈ J, define $k_i = |\{j : i = i(j)\}| = \bigcup_{j \in J} 1_{i = i(j)}$ be the number of times that the i’th independent set $l_i$ is used in the mapping $i : [m] \to J$. 

$\bigcup_{j \in J} 1_{i = i(j)}$
Augmenting path theorem

**Theorem 4.1**

*If there is a directed path from s to t in G, then there exists y′ ∈ P with y ≤ y′ ≤ x, with y′(E) > y(E). If there is no such path, then there exists a set A ⊆ E s.t. y(A) = r(A) and y(E \ A) = x(E \ A).*

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Let $\delta > 0$ be small.
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If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

**Proof of Thm 4.1.**

15 Now, for $i \in J$, define $k_i = |\{j : i = i(j)\}| = \bigcup_{j \in J} 1_{i = i(j)}$ be the number of times that the $i$'th independent set $I_i$ is used in the mapping $i : [m] \rightarrow J$.

16 Let $\delta > 0$ be small.

17 Update $\lambda$: For each $i \in J$, set $\lambda_i \leftarrow \lambda_i - k_i \delta$. So now, $\sum_i \lambda_i = 1 - m\delta$. 

$\sum_i k_i = m$
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*If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.*

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15. Now, for $i \in J$, define $k_i = |\{j : i = i(j)\}| = \bigcup_{j \in J} 1_{i = i(j)}$ be the number of times that the $i$'th independent set $I_i$ is used in the mapping $i : [m] \to J$.

16. Let $\delta > 0$ be small.

17. Update $\lambda$: For each $i \in J$, set $\lambda_i \leftarrow \lambda_i - k_i \delta$. So now, $\sum_i \lambda_i = 1 - m \delta$. So, $\lambda$ is a valid. Note (the weight).

18. Next, we add $e_1$ to $\bigcup_i I_i$, and distribute amongst them to remove any circuits, as follows.
Augmenting path theorem

**Theorem 4.1**

If there is a directed path from \( s \) to \( t \) in \( G \), then there exists \( y' \in P \) with \( y \leq y' \leq x \), with \( y'(E) > y(E) \). If there is no such path, then there exists a set \( A \subseteq E \) s.t. \( y(A) = r(A) \) and \( y(E \setminus A) = x(E \setminus A) \).

**Proof of Thm 4.1.**

1. for \( j \in 1 \ldots m - 1 \) do
2. \[
    l_{i(j)} \leftarrow l_{i(j)} + e_j - e_{j+1} \quad \text{and} \quad \lambda_{i(j)} \leftarrow \lambda_{i(j)} + \delta;
\]
3. \[
    l_{i(m)} \leftarrow l_{i(m)} + e_m \quad \text{and} \quad \lambda_{i(m)} \leftarrow \lambda_{i(m)} + \delta;
\]

Due to how \( G \) is constructed, the updated \( I_i \) are all still independent (we add \( e_j \) to \( I_i(j) \) creating a circuit broken by removing \( e_j+1 \)).

Also, we again have \( \sum_i \lambda_i = 1 \). Choose \( \delta > 0 \) small enough for \( \lambda_i \geq 0 \), and that \( y' = \sum_{i \in J} \lambda_i I_i = y + \delta \).

The theorem is proven.
Augmenting path theorem

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If there is a directed path from $s$ to $t$ in $G$, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

**Proof of Thm 4.1.**

$$
\begin{align*}
1 & \text{ for } j \in 1 \ldots m - 1 \text{ do} \\
2 & \quad I_i(j) \leftarrow I_i(j) + e_j - e_{j+1} \text{ and } \lambda_i(j) \leftarrow \lambda_i(j) + \delta \; ; \\
3 & \quad I_i(m) \leftarrow I_i(m) + e_m \; ; \text{ and } \lambda_i(m) \leftarrow \lambda_i(m) + \delta \; ;
\end{align*}
$$

Due to how $G$ is constructed, the updated $l_i$ are all still independent (we add $e_j$ to $l_i(j)$ creating a circuit broken by removing $e_{j+1}$).
Theorem 4.1

If there is a directed path from s to t in G, then there exists $y' \in P$ with $y \leq y' \leq x$, with $y'(E) > y(E)$. If there is no such path, then there exists a set $A \subseteq E$ s.t. $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$.

Proof of Thm 4.1.

1. **for** $j \in 1 \ldots m - 1$ **do**
   2. $I_i(j) \leftarrow I_i(j) + e_j - e_{j+1}$ and $\lambda_i(j) \leftarrow \lambda_i(j) + \delta$;
   3. $I_i(m) \leftarrow I_i(m) + e_m$; and $\lambda_i(m) \leftarrow \lambda_i(m) + \delta$;

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Augmenting path theorem

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```plaintext
for $j \in 1 \ldots m - 1$ do
  $l_{i(j)} \leftarrow l_{i(j)} + e_j - e_{j+1}$ and $\lambda_{i(j)} \leftarrow \lambda_{i(j)} + \delta$;
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21 The theorem is proven.
Augmenting path theorem consequences

**Corollary 4.2**

*For any $x \in \mathbb{R}^E_+$, we have*

$$\max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (12)$$

*Note: this was not used in the theorem above, rather it is a consequence!*

**Proof.**

1. First, any $y \in P$ with $y \leq x$, and any $A \subseteq E$, we have

$$y(E) = y(A) + y(E \setminus A) \leq r(A) + x(E \setminus A) \quad (13)$$

as we have seen.
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For any \( x \in \mathbb{R}^E_+ \), we have

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2. So we need only find a \( y \) giving equality.
Augmenting path theorem consequences

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4. Then there exists no such \( y' \in P \) s.t. \( y'(E) > y(E) \), and the digraph won’t have a directed path from \( s \) to \( t \) (by the theorem).
**Augmenting path theorem consequences**

**Corollary 4.2**

*For any* $x \in \mathbb{R}^E_+$, *we have*

$$\max (\ y(E) : y \leq x, y \in P_f ) = \min ( \ x(A) + f(E \setminus A) : A \subseteq E ) \quad (12)$$

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**Proof.**

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5. Then, there is a set $A$ such that $y(A) = r(A)$ and $y(E \setminus A) = x(E \setminus A)$, or that $y(E) = r(A) + x(E \setminus A)$, thus demonstrating equality.
Corollary 4.3

Given matroid $M$, we have

$$P_{ind.\ set} = P_r \quad (14)$$

We even get this a consequence!

Proof.

- We saw before (in lecture 7) that this follows from corollary 4.2 (which we encountered in lecture 7).
Corollary 4.3

Given matroid $M$, we have

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Proof.

- We saw before (in lecture 7) that this follows from corollary 4.2 (which we encountered in lecture 7).
- Therefore, the equivalence follows indirectly just from Theorem 4.1!!
Augmenting paths

- The above theorem clearly defines an algorithm, but the question is how efficient it is.
Augmenting paths

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- An analysis similar to the augmenting-path analysis of Edmonds/Karp for max-flow can be used here as well, and we’ll outline these steps (details in Cunningham-84).
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- Key in the max-flow result (to achieve polynomial time) is to always use the shortest path possible (so always use shortcut free paths, ones for which no shortcut exists).
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- To find a short-cut free path, we “scan” and “label” nodes, where a node is scanned by examining all incident edges, and labels are given to any previously unlabeled adjacent nodes.
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- Key in this is to: 1) scan nodes in the order that they are labeled, and 2) label nodes (from a node being scanned) in an order consistent with some fixed total order on all vertices.

While 1) ensures that the path has as few edges as possible (proven in Edmonds/Karp), 2) results in a lexicographically minimum order. Both together are called a consistent breadth-first search, or CBFS.
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  2. The only way an edge becomes available for use in an augmenting path is by being used in the opposite direction in the previous augmentation.
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  1. In every path used for augmentation, there is one critical edge, that becomes unavailable for use immediately after the augmentation.
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• On our current context, we have results quite similar to this that guarantee that the number of augmentations is polynomially bounded, yielding our next theorem.
Bounding the number of augmenting paths

- Consider the algorithm implied by Theorem 4.1 as producing one augmentation, and let $G_i$ refer to the digraph at outer iteration $i$. Then we have
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**Theorem 4.4**

Let $G_0, G_1, \ldots, G_k$ be a sequence of digraphs, each having vertex set $E \cup \{s, t\}$, and correspond to such graphs each one running the algorithm implied by theorem 4.1 Assume fixed total order of $E \cup \{s\}$. Let $Q_i$ denote the CBFS path in $G_i$, for $0 \leq i < k$. If it is the case that, for $0 \leq i < k$:

1. There is an edge in $Q_i$ that is not an edge in $G_{i+1}$,
2. If $(e, f)$ is an edge in $G_{i+1}$ but not in $G_i$, then $e, f \in E$ and there are vertices $a, b \in Q_i$ with $a$ preceding $b$ on $Q_i$ such that: 1) either $a = f$ or $(a, f)$ is an edge in $G_i$; and 2) $b = e$ or $(e, b)$ is an edge in $G_i$,

Then we have that the number of augmentations has bound $k \leq |E|^3$. 

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Theorem 4.5

It is possible to construct an augmentation scheme such that each augmenting path is done in accordance to Theorem 4.4. Each such augmentation is CBFS, and is called a “grand” augmentation, and is maximal in a certain way. This achieves the $O(n^3)$ time, in the number of augmentations, mentioned above.
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- Of course, the cost of each augmentation might be expensive. For matric matroids, each would be $O(r^2n^5)$ where $r$ is the number of rows of the matrix, leading to $O(r^2n^8)$ algorithm.
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- Of course, the cost of each augmentation might be expensive. For matric matroids, each would be $O(r^2n^5)$ where $r$ is the number of rows of the matrix, leading to $O(r^2n^8)$ algorithm.
- On the other hand, this algorithm has some intriguing properties.
Recall the Edmonds matroid partition algorithm, was SFM for \( r(A) - \frac{1}{k} \mathbf{1}(A) \).
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We now have an algorithm that can do $r(A) - x(A)$ for any $x \in \mathbb{R}^E_+$. 

There are three limitations to this:

1. $r(A)$ is only a matroid rank function (and thus integral) rather than a (possibly non-integral) polymatroidal function.
2. $x$ is required to be positive $x \geq 0$.
3. This works only for the difference between $r$ and $x$, but we'd like an algorithm that works for any arbitrary submodular function $f$, even non-monotone and/or non-non-increasing/decreasing.

It turns out that (2) and (3) is easy to deal with, but (1) took another 16 years to solve (and perhaps can still be seen as unsolved, w.r.t. wanting a scalable algorithm).
Recall the Edmonds matroid partition algorithm, was SFM for $r(A) - \frac{1}{k} 1(A)$.

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Towards SFM

- First, given any submodular function $g$, construct modular function $m : E \rightarrow \mathbb{R}$ such that $m(e) = g(E \setminus \{e\}) - g(E)$. 

$$f(A) = g(A) + m(A) - g(\emptyset)$$

Then $f(\emptyset) = 0$, so $f$ is normalized. Also, $f$ is monotone non-decreasing and submodular. It is submodular since sum of submodular and modular. Monotone non-decreasing follows since

$$f(B + v) - f(B) = g(B + v) - g(B) + m(v)$$

$$\geq 0$$

since, by submodularity,$$g(B + v) - g(B) \geq g(E - v) - g(E).$$
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- Also, given $f(A) = g(A) + m(A) - g(\emptyset)$, to minimize $g$ we can just minimize $f - m$. 

\[ 	ext{Is } m \in \mathbb{R}^E^+? \]

No, but for any $e$ such that $m(e) < 0$ can't be a minimizer of $f$ since, assuming that $A$ minimizes $f(A) - m(A)$ and $e \in A$ is such that $m(e) < 0$, then we have that $f(A') - m(A') < f(A) - m(A)$ where $A' = A \{ e \}$. This follows since $f$ is monotone non-decreasing, and $m(A) = m(A') + m(e)$. This deals with (2) above.

Therefore, SFM is as "easy" as moving from matroid rank functions to not-necessarily-integral polymatroidal functions.
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- Also, given \( f(A) = g(A) + m(A) - g(\emptyset) \), to minimize \( g \) we can just minimize \( f - m \).

- So now we have a difference of a polymatroid function and a modular function. This deals with (3) above.
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- Therefore, SFM is as “easy” as moving from matroid rank functions to not-necessarily-integral polymatroidal functions.
Sources for Today’s Lecture

- W. Cunningham, “Testing Membership in Matroid Polyhedra”, 1984