6.1 Properties of ROC (continued)

More on Property 5

$x[n]$ is a right sided sequence and $x[n] = 0$ for $n < N_1$. Let $r_0$ such that

$$\sum_{n=-\infty}^{\infty} |x[n]| r_0^{-n} = \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty$$

So

$$\{z : |z| = r_0\} \subset \text{ROC}$$

Now see if

$$\{z : |z| > r_0\} \subset \text{ROC}$$

Consider $r_1 \geq r_0$ and let $r_1 = cr_0$ for $c \geq 1$.

$$\sum_{n=N_1}^{\infty} |x[n]| r_1^{-n}$$

$$= \sum_{n=N_1}^{\infty} |x[n]| (cr_0)^{-n}$$

$$= \sum_{n=N_1}^{-1} |x[n]| c^{-n} r_0^{-n} + \sum_{n=0}^{\infty} |x[n]| c^{-n} r_0^{-n}$$

$$\leq c^{N_1} \sum_{n=N_1}^{-1} |x[n]| r_0^{-n} + c^{N_1} \sum_{n=0}^{\infty} |x[n]| r_0^{-n}$$

$$= A \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty \quad (\therefore \{z : |z| = r_0\} \subset \text{ROC})$$

where $A = \left( \frac{r_1}{r_0} \right)^{N_1}$. So

$$\sum_{n=N_1}^{\infty} |x[n]| r_1^{-n} < \infty$$

which means

$$\{z : |z| > r_0\} \subset \text{ROC}$$

Property 7. If $x[n]$ is two sided sequence, and if the circle $|z| = r_0$ is in ROC, then the ROC will consist of a ring in the $z$-plane which includes $|z| = r_0$.

Why?

$$x[n] = x_L[n] + x_R[n]$$
where $x_L[n]$ is a left sided sequence and $x_R[n]$ is a right sided sequence. ROC for $x_R[n]$ is bounded inside by outmost pole of $X_R(z)$. ROC for $x_L[n]$ is bounded outside by innermost pole of $X_L(z)$. Add the two together. Either the two ROC’s overlap, which is a ring shaped ROC for $x[n]$, or there is no ROC for $x[n]$.

**Property 8.** ROC must be a connected region.

Note from Property 7, any sequence is the sum of a left sided sequence and a right sided sequence. So if intersection exists, it must be formed by

$$d_R < |z| < d_L$$

Hence, poles determine possible types of ROC’s.

**Ex:** When there are two poles $d_1$ and $d_2$, and $|d_1| < |d_2|$, there are three possible ROC candidates, i.e.

$$ROC_1 = \{z : |z| < |d_1|\}$$
$$ROC_2 = \{z : |d_1| < |z| < |d_2|\}$$
$$ROC_3 = \{z : |d_2| < |z|\}$$

Figure 6.1: A $z$-transform $H(z) = \frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.82z^{-1}+0.64z^{-2}}$ with poles around $0.4 \pm j0.6928$ and zeros around $1.2 \pm j1.2$

### 6.2 Inverse $z$-Transform

$z$-Transform is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Let $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} = X(re^{j\omega}) = FT\{x[n]r^{-n}\}$$
So
\[ \text{FT}^{-1} \{ X(re^{j\omega}) \} = x[n]r^{-n} \]

Then
\[
x[n] = r^n \text{FT}^{-1} \{ X(re^{j\omega}) \}
  = r^n \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})e^{j\omega n} d\omega
  = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n d\omega
\]

Since \( z = re^{j\omega} \),
\[
dz = jre^{j\omega}d\omega = jzd\omega
\]
that is,
\[
d\omega = \frac{1}{j} z^{-1}dz
\]

Therefore,
\[
x[n] = \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(z)z^{n-1}dz
\]
But \( r \) can be anything inside the ROC. The inverse \( z \)-transform equation can be written as
\[
x[n] = \frac{1}{2\pi j} \oint_{c} X(z)z^{n-1}dz
\]
(6.1)

The contour integral \( \oint_{c} \) is performed at a counter-clockwise closed circular contour centered at the origin with radius \( r \), for all \( r \) such that
\[
\{ z : |z| = r \} \subset \text{ROC}
\]

Another way to derive this.

**Theorem 6.1 (Cauchy Integral Theorem).**

\[
\frac{1}{2\pi j} \oint_{c} z^{-k}dz = \begin{cases} 
1 & k = 1 \\
0 & k \neq 1 
\end{cases}
\]
(6.2)


Consider
\[
\frac{1}{2\pi j} \oint_{c} X(z)z^{n-1}dz = \frac{1}{2\pi j} \oint_{c} \sum_{k=-\infty}^{\infty} x[k]z^{-k+n-1}dz
\]
\[
= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi j} \oint_{c} z^{-k+n-1}dz
\]
\[
= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad (\because \text{Cauchy Integral Theorem})
\]
\[
= x[n]
\]

Note: the result is valid for both positive and negative values of \( n \).

Note: for \( z = e^{j\omega} \), this reduces to inverse FT, since
\[
\frac{1}{2\pi j} \oint_{c} X(z)z^{n-1}dz \bigg|_{z = e^{j\omega}} = \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}e^{-j\omega}j e^{j\omega}d\omega
\]
since \( dz = jre^{j\omega}d\omega \).

How to calculate this in general?
Theorem 6.2 (Cauchy Residue Theorem).

\[ x[n] = \frac{1}{2\pi j} \oint_{c} X(z)z^{n-1}dz = \sum \text{[residues of } X(z)z^{n-1} \text{ at poles inside } c] \] (6.3)


Finding residues (to be defined shortly) is often difficult. But if \( X(z)z^{n-1} \) is a rational function of \( z \),

\[ X(z)z^{n-1} = \frac{\Psi(z)}{(z-a)^{s}} \] (6.4)

If \( X(z)z^{n-1} \) has \( s \) poles at \( z = d_{0} \) and \( \Psi(z) \) has no poles at \( z = d_{0} \), then

\[ \text{Residue } X(z)z^{n-1} \text{ at } z = d_{0} = \frac{1}{(s-1)!} \left[ \frac{d^{s-1}\Psi(z)}{dz^{s-1}} \right]_{z=d_{0}} \] (6.5)

So if \( s = 1 \) (1st order pole), then

\[ \text{Residue } X(z)z^{n-1} \text{ at } z = d_{0} = \Psi(d_{0}) \]

**Ex:** Find the inverse z-transform of

\[ X(z) = \frac{1}{1-az^{-1}} \quad |z| > |a| \]

\[ x[n] = \frac{1}{2\pi j} \oint_{c} \frac{z^{n-1}}{1-az}dz = \frac{1}{2\pi j} \oint_{c} \frac{z^{n}}{z-a}dz \]

The radius of contour must be greater than \( a \) since the contour has to be in ROC. We will consider the contour integral for \( n \geq 0 \) and \( n < 0 \) separately.

1. \( n \geq 0 \)
   
   There is only one pole at \( z = a \), i.e.,

   \[ \frac{z^{n}}{z-a} \text{ has only one pole at } z = a \]

   So \( \Psi(z) = z^{n} \) and \( \Psi(a) = a^{n} \). Therefore

   \[ x[n] = a^{n} \quad n \geq 0 \]

   since \( s = 1 \) and no derivative is necessary.

2. \( n < 0 \)
   
   There are multiple order poles at \( z = 0 \), with the order depending on \( n \). There is one pole at \( z = a \).

   When \( n = -1 \), there are a 1st order pole at \( z = 0 \) and a 1st order pole at \( z = a \).

   \[ \text{Residue } [X(z)z^{n-1}]_{n=-1} \text{ at } z = a = a^{-1} \quad (\therefore \Psi(z) = z^{n}) \]

   \[ \text{Residue } [X(z)z^{n-1}]_{n=-1} \text{ at } z = 0 = -a^{-1} \quad (\therefore \Psi(z) = \frac{1}{z-a}) \]

   So residues sum to 0, i.e.,

   \[ x[-1] = 0 \]

   When \( n = -2 \), there are two poles at \( z = 0 \) and one pole at \( z = a \).

   \[ X(z)z^{n-1} = \frac{z^{-2}}{z-a} = \frac{\Psi_{1}(z)}{(z-a)^{1}} = \frac{\Psi_{2}(z)}{(z-0)^{2}} \]

   with

   \[ \Psi_{1}(z) = z^{-2} \text{ and } \Psi_{2}(z) = (z-a)^{-1} \]
So

(Pole at \( z = a, s = 1 \)) Residue \( [X(z)z^{n-1}|_{n=-2} \text{ at } z = a] = \Psi_1(a) = a^{-2} \)

(Pole at \( z = 0, s = 2 \)) Residue \( [X(z)z^{n-1}|_{n=-2} \text{ at } z = 0] = \frac{d}{dz}\Psi_2(z)\bigg|_{z=0} = \frac{d}{dz}(z-a)^{-1}\bigg|_{z=0} = (z-a)^{-2}|_{z=0} = -a^{-2} \)

Residues cancel out again, so \( x[-2] = 0 \)

We can continue this for \( x[-3], x[-4], \ldots \)

**Easier Ways to Compute Inverse \( z \)-Transform**

**Inspection.**

Use \( z \)-transform table.

Ex:

\[
\frac{a^n u[n]}{1 - az^{-1}} \quad |z| > |a|
\]

So

\[
\frac{1}{1 - \frac{2}{3}z^{-1}} z \rightarrow \left(\frac{2}{3}\right)^n u[n] = x[n]
\]

if \( x[n] \) is a right sided sequence.

Ex:

\[
X(z) = \frac{1}{1 - a_1z^{-1}} + \frac{1}{1 - a_2z^{-1}} \quad a_1 < |z| < a_2
\]

so

\[
x[n] = a_1^n u[n] - a_2^n u[-n-1]
\]

since ROCs of each of the terms in \( X(z) \) must overlap for the existence of \( X(z) \).

**Partial Fraction Expansion.**

(There are some nice forms of \( z \)-transform that we can get their inverse relatively easily.)

When

\[
X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad (6.6)
\]

We can factor the denominator and numerator and get

\[
X(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})} \quad (6.7)
\]

1. \( N > M \) and the poles are all first order poles (i.e., all the poles are unique).

\[
X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad (6.8)
\]

Note:

\[
A_k = (1 - d_k z^{-1})X(z)|_{z=d_k} \quad (6.9)
\]

since

\[
(1 - d_k z^{-1})X(z) = A_k + \sum_{l \neq k, l=1}^{N} A_l (1 - d_l z^{-1}) \frac{1}{1 - d_l z^{-1}}
\]
Ex: \[ X(z) = \frac{2}{1 - \frac{3}{2}z^{-1} + \frac{1}{4}z^{-2}} = \frac{2}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - \frac{1}{4}z^{-1}} \]

and

\[ A_1 = (1 - \frac{1}{2}z^{-1})X(z) \big|_{z=\frac{1}{2}} = 4 \]
\[ A_2 = (1 - \frac{1}{4}z^{-1})X(z) \big|_{z=\frac{1}{4}} = -2 \]

So

\[ X(z) = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{2}{1 - \frac{1}{4}z^{-1}} \quad \rightarrow \quad x[n] = 4 \left( \frac{1}{2} \right)^n u[n] - 2 \left( \frac{1}{4} \right)^n u[n] \]

2. \( N \leq M \) and the poles are all first order poles (i.e., all the poles are unique).

\[ X(z) = \sum_{k=0}^{N} b_k z^{-k} = \sum_{r=0}^{N} B_r z^{-r} \sum_{k=1}^{M} A_k \]

(6.10)

The first term is obtainable by long division (example later). The second term is obtainable by the same procedure used when \( N > M \).

In general

\[ X(z) = \sum_{r=0}^{\max(M-N,0)} B_r z^{-r} + \sum_{k=0}^{M-N} b_k z^{-k} \]

(6.11)

\[ \sum_{k=0}^{\max(M-N,0)} \sum_{i=1}^{N_i} A_k \]

\[ \frac{1}{1-d_k z^{-1}} \]

where

\[ N_i \triangleq \# \text{ of unique poles in denominator} \]
\[ S_i = \text{order of } i\text{-th pole} \]
\[ B_r \text{ is obtained as before} \]

\[ A_{dk} = \frac{1}{(s_i-k)!(-d_i)^{s_i-k}} \left[ \frac{d^{s_i-k}}{dw^{s_i-k}} \left( (1-d_i w)^{s_i} X(w^{-1}) \right) \right]_{w=d_i^{-1}} \]

(6.12)

We can verify this by multiplying both sides of \( X(z) \) by \((1-d_i z^{-1})^0\) and following the above operations.

**Power Series Expansion.**

Ex:

\[ X(z) = z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1} \]

then

\[ x[n] = \delta[n+2] - \frac{1}{2} \delta[n+1] - \delta[n] + \frac{1}{2} \delta[n-1] \]

**Ex:** Power series.

\[ X(z) = \log(1 + az^{-1}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n z^{-n}}{n} \quad \rightarrow \quad x[n] = \begin{cases} (-1)^{n+1} a^n & n \geq 1 \\ 0 & n \leq 0 \end{cases} \]

**Long Division.**

\[ X(z) = \frac{1 + 2.0 z^{-1}}{1 + 0.4 z^{-1} - 0.12 z^{-2}} 
= 1 + 1.6 z^{-1} - 0.52 z^{-2} + 0.4 z^{-3} + \cdots \]
Then

\[ x[n] = 0 \quad n < 0 \]
\[ x[0] = 1 \]
\[ x[1] = 1.6 \]
\[ x[2] = -0.52 \]
\[ x[3] = 0.4 \]
\[ \vdots \]