5.1 Convergence of the $z$-Transform

$X(z)$ might not converge for all $z$. For uniform convergence, absolute summability requires

$$\sum_{n=-\infty}^{\infty} |x[n]|z^{-n} < \infty$$

(5.1)

$z$-transform might converge even if FT does not.

**Ex:** $u[n]$ doesn’t converge with $|z| = 1$, since

$$\sum_{n=0}^{\infty} e^{-j\omega n}$$

doesn’t converge. But

$$\sum_{n=0}^{\infty} r^{-n}e^{-j\omega n}$$

converges if $r > 1$.

**Definition 5.1 (Region of Convergence or ROC).**

$$ROC \triangleq \{ z : \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty \}$$

(5.2)

i.e., locations of absolute summability of the sequence.

So, if $z_1 \in \text{ROC}$, then $|z_1|e^{-j\omega} \in \text{ROC}, \forall \omega$. (since ROC depends only on magnitude.) For the signal we work with, ROC is a disk in $z$-plane, as shown in Fig 5.1. FT exists iff (if and only if) the ROC contains unit circle $|z| = 1$. Uniform
convergence requires absolute summability of \( x[n]z^{-n} \). Note we normally think of FT as evaluation of \( z \)-transform at \( |z| = 1 \).

Exceptions:

\[
x_1[n] = \frac{\sin \omega_c n}{n}
\]
is not absolutely summable even multiplied by \( z^{-n} \), but is square summable.

\[
x_2[n] = \cos \omega_0 n
\]
is not even square summable even multiplied by \( z^{-n} \). So it is not strictly true that FT’s of a sequence corresponds to its \( z \)-transform on unit circle for all sequences.

Recall:

\[
FT\{x_1[n]\} = \begin{cases} 1 & |\omega| < \omega_c, \\
0 & \omega_c \leq |\omega| \leq \pi
\end{cases}
\]

and

\[
FT\{x_2[n]\} = \sum_{k=-\infty}^{\infty} \pi \delta(\omega - \omega_0 + 2k\pi) + \pi \delta(\omega + \omega_0 + 2k\pi)
\]

are neither differentiable or continuous. Note

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \lim_{M\to\infty} \sum_{n=-M}^{M} x[n]z^{-n}
\]
is a polynomial in \( z \), so it should be continuous and differentiable. So \( X(z) \) can not converge to something else that is not continuous or differentiable, i.e., \( X_M(z) \) does not converge to a non-continuous function of \( z \). Nevertheless, we make an exception in this case as well, and allow the FT to exist for \( x_1[n] \) and \( x_2[n] \).

**Definition 5.2 (Rational \( z \)-Transform).** A \( z \)-transform, when it can be summed and expressed in the simple form:

\[
X(z) = \frac{P(z)}{Q(z)}
\]
is called rational \( z \)-transform. \( P(z) \) and \( Q(z) \) are polynomials in \( z \).

Note: this form can be used to represent sums of (complex) exponentials (which are what we study in this class.) Why? (See examples.) \( z \)-transform is linear.

*Zeros of \( X(z) \) are*

\[
\{ z : X(z) = 0 \} \quad (5.3)
\]

*Poles of \( X(z) \) are*

\[
\{ z : X(z) = \infty \} \quad (5.4)
\]

**Definition 5.3 (Right Sided Sequence).** \( x[n] \) is right sided if \( \exists N \) s.t. \( x[n] = 0 \) for \( n < N \).

**Questions:** Is causal sequence right sided? Yes. Is right sided sequence causal? No.

**Definition 5.4 (Left Sided Sequence).** \( x[n] \) is left sided if \( \exists N \) s.t. \( x[n] = 0 \) for \( n > N \).

**Definition 5.5 (Two Sided Sequence).** If \( x[n] \) is neither left sided nor right sided, then \( x[n] \) is two sided.

**Ex 1:** \( x[n] = a^nu[n] \). Note \( a \) could be complex \( a^n = |a|^n e^{j\theta n} \) which includes sinusoids.

\[
X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

Convergence requires

\[
\sum_{n=0}^{\infty} |az^{-1}|^n < \infty
\]
or
\[ |z| > |a| \]
So
\[ X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad |z| > |a| \]
There is a pole at \( z = a \) and a zero at \( z = 0 \). But at \( z = 0 \), the \( z \)-transform does not converge.

**Ex 2:**
\[ x[n] = -a^n u[-n - 1] \]
which is a left sided sequence.
\[ X(z) = -\sum_{n=-\infty}^{0} a^n z^{-n} = -\sum_{k=1}^{\infty} a^{-k} z^k = 1 - \sum_{k=0}^{\infty} (a^{-1} z)^k \]
Convergence requires \(|a^{-1} z| < 1\), i.e., \(|z| < |a|\).

Note: both examples have same pole-zero diagram with pole at \( z = a \). So the same \( z \)-transform can lead to two different sequences. Therefore it is necessary to give both \( z \)-transform and ROC to uniquely specify a sequence.

**Ex:**
\[ x[n] = \left( \frac{1}{2} \right)^n u[n] + \left( -\frac{1}{3} \right)^n u[n] \]
\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2} z^{-1} \right)^n + \sum_{n=0}^{\infty} \left( -\frac{1}{3} z^{-1} \right)^n = \frac{2(1 - \frac{1}{7} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} \]
The ROC is \( \{ |z| > \frac{1}{2} \} \cap \{ |z| > \frac{1}{3} \} = \{ |z| > \frac{1}{2} \} \)

**Ex:** Two-sided sequence
\[ x[n] = \left( -\frac{1}{3} \right)^n u[n] - \left( \frac{1}{2} \right)^n u[-n - 1] \]
\[ X(z) = \frac{2(1 - \frac{1}{2} z^{-1})}{(1 + \frac{1}{2} z^{-1})(1 - \frac{1}{2} z^{-1})} \]
The intersection between the ROC of the first term, i.e., \(|z| > \frac{1}{3}\), and the ROC of the second term, i.e., \(|z| < \frac{1}{2}\), gives the ROC of \( X(z) \), i.e., \( \frac{1}{3} < |z| < \frac{1}{2} \).

**Ex:** Finite Length Sequence
\[ x[n] = a^n u[n] - a^n u[n - N] = \begin{cases} a^n & 0 \leq n \leq N - 1 \\ 0 & \text{else} \end{cases} \]
\[ X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{z^N - a^N}{z^N-1(z-a)} \]
So ROC = \( \{ z: z \neq 0 \} \). Note: pole-zero cancellation occurred by adding two sequences because the roots of \( z^n - a^n = 0 \) are \( z = e^{-j2\pi k/N} \) for \( k = 0, 1, 2, \cdots, N - 1 \). The \( z = a \) root is cancelled by the \( z - a \) term in the denominator.

Note: [\( \cos(\omega_0 n) \)] is right sided sequences converge for \(|z| > 1\), but FT doesn’t exist in strict uniform convergence sense.

(As per O&S Table 3.1)
Table 5.1: Common z-Transform Pairs

<table>
<thead>
<tr>
<th>Sequence</th>
<th>z-Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ[n]</td>
<td>1</td>
<td>All z</td>
</tr>
<tr>
<td>a^n u[n]</td>
<td>1 - a^{-n}</td>
<td></td>
</tr>
<tr>
<td>-a^n u[-n-1]</td>
<td>1 - a^{-n}</td>
<td></td>
</tr>
<tr>
<td>[cos(ω_0 n)] u[n]</td>
<td>(\frac{1 - (\cos ω_0) z^{-1}}{1 - 2 \cos ω_0 z^{-1} + z^{-2}})</td>
<td></td>
</tr>
<tr>
<td>a^n u[n] - a^n u[n-N]</td>
<td>(\frac{1 - a^{-N} z^{-N}}{1 - a^{-n} z^{-n}})</td>
<td></td>
</tr>
</tbody>
</table>

5.2 Properties of ROC for the z-Transform

Property 1. ROC is a ring or a dish in z-plane centered at origin, i.e.,

\[0 ≤ r_R < |z| < r_L ≤ ∞\]

Why? If \(z_1 \in ROC\), so is \(|z_1| e^{jω} \in ROC\) for all \(ω\). So ROC has a ring or dish shape.

Property 2. FT of \(x[n]\) converges absolutely iff

\(\{z : |z| = 1\} \subset ROC\)

Why?

\(X(e^{jω}) = X(z)|_{z = e^{jω}}\)

Property 3. ROC contains no poles by definition.

Property 4. If \(x[n]\) is finite duration sequence, i.e., is non-zero only when

\(-∞ < N_1 ≤ n ≤ N_2 < ∞\)

ROC will be either \(\{z\}\) or \(\{z \neq 0\}\) or \(\{z \neq ∞\}\) or \(\{z \neq 0\) and \(z \neq ∞\}\).

Why?

\(X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}\)

Convergence requires \(\sum_{n=N_1}^{N_2} |x[n]| |z|^{-n} < ∞\). We need \(|x[n]| < ∞\) and the rest is a finite sum. Possible problems:

If \(N_2 > 0\), poles will be at \(z = 0\).

If \(N_1 < 0\), poles will be at \(z = ∞\).

If \(N_1 = N_2 = 0\), no poles or zeros at all.

Ex: \(δ[n] \xrightarrow{z} 1\).

Property 5. If \(x[n]\) is a right sided sequence, then ROC extends outward from outmost (largest magnitude) finite pole of \(X(z)\) to (and possibly including) \(z = ∞\).

Why? Consider

\(X(z) = \sum_{n=N_1}^{∞} x[n]z^{-n}\)

Suppose

\(z_1 = r_1 e^{-jω_1} \in ROC\)
then
\[ z = |z_1|e^{-j\omega} \in \text{ROC} \quad \forall \omega \]

**Question:** is \( z_2 = r_2e^{-j\omega_2} \) (with \( r_2 > r_1 \)) in ROC?

Consider \( N_1 < 0 \). If \( n < 0 \), everything is still finite, so it is ok. For \( n > 0 \), if
\[
\sum_{n=0}^{\infty} |x[n]| |r_1|^{-n}
\]
converges, so does
\[
\sum_{n=0}^{\infty} |x[n]| |r_2|^{-n}
\]
converge if \( r_2 > r_1 \) since it decays only faster. Note this occurs at outmost pole and also \( z = \infty \in \text{ROC} \) if \( N_1 > 0 \).

We will see that for \( n \geq N_1 \), we can write
\[
x[n] = \sum_{k=1}^{N_1} A_k d_k^n \quad n \geq N_1
\]
For convergence, each term in the following sum should absolutely summable,
\[
x[n] r^{-n} = \sum_{k=1}^{N_1} A_k (d_k r^{-1})^n \quad n \geq N_1
\]
i.e.,
\[
\sum_{n=N_1}^{\infty} |d_k r^{-1}|^{-n} < \infty \quad \forall k
\]
Therefore,
\[
|r| > |d_k| \quad \forall k
\]
That is,
\[
|r| > \max_k |d_k|
\]

**Property 6.** If \( x[n] \) is a left sided sequence, its ROC extends inward from the innermost non-zero pole of \( X(z) \) and might include \( z = 0 \).

Why? Same reasoning.

\[
X(z) = \sum_{n=-\infty}^{N_1} x[n] z^{-n}
\]
if \( z_1 \in \text{ROC} \), then \( z_2 \in \text{ROC} \) if \( |z_2| < |z_1| \), since it will decay faster for negative \( n \).

**Property 7.** A two-sided sequence is an infinite duration sequence. If \( x[n] \) is two-sided, ROC will be a ring in \( z \)-plane bounded by magnitude of some of the poles.

Why? Exponential weighting in \( z \) needs to be balanced, so that component right side and left side don’t diverge.