12.1 Decomposable Graphs (continues from last time)

**Theorem 12.1.** The following conditions are equivalent for an undirected graph $\mathcal{G}$:

1. Decomposable graph;
2. Chordal or triangulated graph;
3. Every $(\alpha, \beta)$-separator is complete.

(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) were proved. We will start today’s lecture by proving (3) $\Rightarrow$ (1), or that if every $(\alpha, \beta)$-separator is complete, then it is a chordal (or triangulated) graph. ([SLL1996])

**Proof.** (3) $\Rightarrow$ (1): Assume that every $(\alpha, \beta)$-separator is complete. If $\mathcal{G}$ is complete, then it is decomposable by definition. However, if $\mathcal{G}$ is not complete, it has at least two non-adjacent vertices, $\alpha$ and $\beta$. Let $C$ be the (necessarily complete) minimal $(\alpha, \beta)$-separator which divides $V$ into two disjoint sets 1) the $\alpha$ connected components in $V \setminus C$, 2) $[\alpha]_{V \setminus C}$, the corresponding $\beta$ connected components $[\beta]_{V \setminus C}$, and 3) any other remaining vertices (lets name them $D$). We define the sets $A = [\alpha]_{V \setminus C} \cup D$ and $B = [\beta]_{V \setminus C}$, so that so that $A, B, C$ form a partition in $V$. An example is shown in figure 12.1.

$C$ is complete by assumption. The graph is decomposable if triple $(A, B, C)$ forms a decomposition of $\mathcal{G}$ and if the subgraphs $\mathcal{G}_{A\cup C}$ and $\mathcal{G}_{B\cup C}$ are also decomposable. We already have a decomposition in $(A, B, C)$ since $C$ separates $A$ and $B$ and $C$ is complete. To show that the subgraphs are also decomposable, we use induction.

Let $C_1$ be the minimal $(\alpha_1, \beta_1)$ separator in $\mathcal{G}_{A\cup C}$ for some $\alpha_1, \beta_1 \in A \cup C$. If $C_1$ is a minimal $(\alpha_1, \beta_1)$-separator in $\mathcal{G}_{A\cup C}$ (one possible choice of $C_1$ is shown in figure 12.1), it is also the minimal separator in $\mathcal{G}$. This is because no path from $\alpha_1$ to $\beta_1$ would go through $B$ without intersecting $C_1$, since $C$ separates $A$ from $B$ and $C$ is complete. Therefore, $C_1$ is complete by assumption. But since $|V_{A\cup C}| < n$, by induction $\mathcal{G}_{A\cup C}$ is decomposable. A similar argument can be used to show that the subgraph $\mathcal{G}_{B\cup C}$ is also decomposable, and therefore $\mathcal{G}$ is decomposable.
Finally, the equivalence relation stated in theorem 12.1 is proved.

12.2 Junction Trees

Now, we shall revisit the definition of junction tree and examine how it is related to the family of decomposable graphs.

**Definition 12.2.** Let $\mathcal{C}$ be a collection of subsets of a finite set $V$ and $\mathcal{T}$ a tree with $\mathcal{C}$ as its node set. Then $\mathcal{T}$ is said to be a junction tree if any intersection $C_1 \cap C_2$ of a pair $C_1, C_2$ of sets in $\mathcal{C}$ is contained in every node on the unique path in $\mathcal{T}$ between $C_1$ and $C_2$.

Junction tree of cliques is a collection $\rho$ (set of subsets of nodes in $V$). Let $\tau$ be a tree with elements of $\rho$ as the nodes. The $\tau$ is a junction tree if any intersection $S_{ij} = C_i \cup C_j, C_i \in \tau, C_j \in \tau,$ or $S_{ij} \subseteq C_k$ is true. Where $C_k$ is any clique on the necessary path between $C_i$ and $C_j$.

![Figure 12.2: Examples of a DAG, its moralized, and triangulated one](image)

Figure 12.2 shows an original graph, the corresponding moralized graph, and one possible triangulation of the moralized graph. Note that the cliques of this last triangulated graph are \{A, B, C, D\}, \{B, C, D, F\}, \{D, F, G\}, \{B, E, F\}, \{F, G, I\}, and \{E, F, H\}.
A junction tree is such that in the graph of cliques (i.e., a graph where the cliques themselves are nodes in the graph), the junction tree is the maximum spanning tree. So in the following graph of cliques, the edges show the counts of the intersections of the cliques. We find the tree with maximum weight to get the junction tree (we will prove this later in the course).

Figure 12.3: Example of the numbers of intersection

Figure 12.3 shows an example of the numbers of intersection, in which the numbers beside each node show the number of each connection. The junction tree is the graph of cliques and is the maximum spanning tree. So figure 12.4 shows a junction tree of the graph in figure 12.3.

Figure 12.4: Example of a junction tree

For some graphs (i.e., decomposable as we will see), junction trees may be constructed using its cliques as illustrated in figure 12.5. Junction trees have the property that probabilistic inference is correct (as we will show later in the course).

Figure 12.5: Example of a junction tree

As shown in figure 12.6, junction trees need not be unique for a graph.
Theorem 12.3. There exists a junction tree \( T \) of cliques for the graph \( \mathcal{G} \) if and only if \( \mathcal{G} \) is decomposable.

This is one of the key theorems in this course. Let's prove it.

Proof. The proof is by induction on number of cliques ([CDLS1999]).

[1] Junction tree of cliques exists \( \Rightarrow \) decomposable: The theorem clearly holds for \( \mathcal{G} \) with two cliques \( C_1 \) and \( C_2 \). In such a case, triple \((A, B, S)\) of decomposable graph consists of \( S = C_1 \cap C_2 \), \( A = C_1 \setminus S \) and \( B = C_2 \setminus S \). \( S \) is complete and separates \( C_1 \setminus S \) with \( C_2 \setminus S \). Suppose that the theorem holds for all graphs with at most \( k \) cliques and let \( \mathcal{G} \) have \( k + 1 \) cliques.

Assume \( T \) is a junction tree of cliques for \( \mathcal{G} \). Take \( C_1 \) and \( C_2 \) adjacent in \( T \). On cutting the link \( C_1 \sim C_2 \), \( T \) separates into two subtrees, \( T_1 \) and \( T_2 \) (since there can only be one path between two nodes in \( T \)). Let \( V_i \) be the union of the nodes in \( T_i \) for \( i = 1, 2 \), and let \( G_i = G_{V_i} \).

[Q] Is \( \mathcal{G} \) decomposable?
The nodes in \( T_i \) are then the cliques of \( G_i \), and \( T_i \) is a junction tree for \( \mathcal{G} \). By the inductive assumption, \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are decomposable. Let \( S = C_1 \cap C_2 \). The triple \((V_1, V_2, S)\) will form a decomposition of \( \mathcal{G} \), if \( S \) is complete and separates \( V_1 \) and \( V_2 \). We need to show

1. \( S \) is complete.;
2. \( S \) separates \( V_1 \) from \( V_2 \);

To prove the first one, let \( v \in S = V_1 \cap V_2 \). There exists a clique \( C_i' \) of \( \mathcal{G}_i \) for each \( i = 1, 2 \), with \( v \in C_i' \). The path from \( C_1' \) and \( C_2' \) passes through \( C_1 \) and \( C_2 \). Therefore, \( v \in C_1 \cap C_2 \Rightarrow V_1 \cap V_2 \subseteq C_1 \cap C_2 \). But, \( C_1 \cap C_2 \subseteq V_1 \cap V_2 \). So, \( S = C_1 \cap C_2 \) and complete.
Now, to prove that $S$ separates $V_1$ and $V_2$, let’s make the assumption that $S$ does not separate $V_1$ from $V_2$. If this is true, then there exists a path $u, w_1, w_2, \ldots, w_k, v$ with each $w_i \notin S$ for all $i$ (i.e., none of the nodes on this path can be in $S$ under this assumption).

Then, there exists a clique $C$ containing the complete set $u, w_1$ with $C \in \mathcal{C}$ where $\mathcal{C}$ is a clique of $\mathcal{T}$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_i$ are cliques of $\mathcal{T}_i$ and where we have $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. However, note that we can’t have $C \in \mathcal{C}_2$ which is equivalent to the condition that $C \subseteq V_2$, since $u \notin V_2$. This implies that $C \in \mathcal{C}_1$ or equivalently that $C \subseteq V_1$. Therefore, $w_1 \in V_1$ and hence $w_1 \in V_1 \setminus S$. Repeating this argument for each $w_i$ along the path, we deduce that $v \in V_1 \setminus S$. This contradicts the assumption. Therefore, $S$ separates $V_1$ and $V_2$. We have already shown that $S$ is complete and from the inductive assumption, we know that the subgraphs from $V_1$ and $V_2$ are decomposable. Therefore, the triple $(V_1, V_2, S)$ forms a decomposition and this completes the proof that every graph $\mathcal{G}$ for which a junction tree of cliques exists is decomposable.

[2] Decomposable $\Rightarrow$ junction tree of cliques exits:
Assume that $\mathcal{G}$ is decomposable, and let $(W_1, W_2, S)$ be a decomposition of $\mathcal{G}$ into proper decomposition subgraph $\mathcal{G}_{W_1}, \mathcal{G}_{W_2}$, where $V_i = W_i \cup S$. Then at least one of $V_1$ and $V_2$ - say $V_1$ has the form $\bigcup_{C \in \mathcal{C}_1} C$ with $\mathcal{C}_1 \subset \mathcal{C}$ and redefining $V_2 = \bigcup_{C \in \mathcal{C}_2} C$ where $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Let $C_i \in \mathcal{C}_i$ satisfy $S \subseteq C_i$. By hypothesis, we have a junction tree for $\mathcal{G}_i$ for $\mathcal{G}_i$, where, as before $\mathcal{G}_i = S_{V_i}$. We form a tree $\mathcal{T}$ by linking $\mathcal{T}_1$ and $\mathcal{T}_2$ via $C_1$ and $C_2$.

Let $v \in V$. If $v \notin V_1$, then all cliques containing $v$ are in $\mathcal{C}_1$, and so connected in $\mathcal{T}_1$, hence in $\mathcal{T}$. If $v \notin V_2$, then all cliques containing $v$ are in $\mathcal{C}_2$, and so connected in $\mathcal{T}_1$, hence in $\mathcal{T}$. If $v \notin V_1$, then similarly for $\mathcal{T}_2$ because it is symmetric. And if $v \in S = V_1 \cap V_2$, then $v \in (C_1 \cap C_2)$. The cliques in $\mathcal{C}_i$ containing $v$ are connected in $\mathcal{T}_i$, and include $C_i$. Since $C_1$ and $C_2$ are connected the result follows.
So, now we have a very important equivalence, decomposable $\equiv$ chordal $\equiv$ every $(\alpha, \beta)$-separator complete $\equiv$ junction tree of cliques exists.

Note that the above proof also shows that an intersection $S = C_1 \cap C_2$ between two neighboring nodes in a junction tree of cliques forms the minimal separator between two decomposable graph $\mathcal{G}$, as illustrated in figure 10.7. As we shall see in later lectures, the cliques and the separators provide meaningful entities in inference computation. The cliques represent the joint (marginal) probability and the separators form the denominator in conditional probability distributions.

[Q] Is $S$ minimal?
Yes. $S$ is a minimal separator.

12.3 Running Intersection Property (r.i.p)

A sequence of sets $(C_1, C_2, \ldots, C_k)$ is said to have the running intersection property if, for all $1 < j \leq k$, there is an $i < j$ such that $C_j \cap (C_1 \cup C_2 \ldots \cup C_{j-1}) \subseteq C_i$.

To interpret along the lines of predictive model, we have

1. $H_j = (C_1 \cup C_2 \ldots \cup C_j)$ represents “history” up to $j$;
2. $C_j \setminus H_{j-1} = R_j$ represents the “residual” or “innovations” in the run of cliques;
3. $S_j = H_{j-1} \cap C_j$ represents the separator set.

The r.i.p. says that $\forall j, S_j \subseteq C_i$ for some $i < j$

![Diagram](image.png)

**Figure 12.13: A property of RIP**

A well-ordered (e.g., breadth-first or depth-first order) ordering of the junction tree cliques is sufficient for the r.i.p.

![Diagram](image.png)

**Figure 12.14: An example of well-ordered junction tree**

### 12.4 Junction Tree Construction Algorithm

We can construct a junction tree from cliques of chordal graph order to satisfy r.i.p.
**Algorithm:** constructing a junction tree

input: $C_1, C_2, \ldots, C_p$ (the cliques of a chorded graph satisfying RIP)

step 1: associate each node in the graph with each clique

step 2: for $j = 2, \ldots, p$

  add an edge between $C_i$ and $C_j$ for some $i < j$ such that $S_j \subseteq C_i$ ($S_j = H_{j-1} \cap C_j$)

For a junction tree, for any node $v$, cliques containing $v$ must induce a connected tree, and there are two cases:

Case 1: If $v \in S_j \subseteq C_i$, then $v \in C_i$ and $v \in C_j$. Therefore it induces a tree which is necessarily connected and containing $v$.

Case 2: If $v \notin S_j$, then $v \in R_j = C_j \setminus H_{j-1}$

The proof of this can be found in Lauritzen’s text.

**References**
