Problem # 1 (20 points)

In this problem, recall that signals can be complex valued. Solve all of the below 4 problems. Each of the problems is worth 5 points. If you answer unknown for any problem, include full justification.

(1a) (5pts) Consider the system described by the x-y-relation as follows:

\[ y(t) = x^2(t) \]  

\[ (0.1) \]

Is this system: i) linear, ii) time-invariant, iii) BIBO stable, iv) invertible, v) memoryless, vi) causal? **Circle yes/no/unknown for each and include brief justification.**

Linear: no, since it doesn’t satisfy scaling. \( \alpha x(t) \) input produces \( \alpha^2 x(t) \) output.

Time-invariant: yes, since \( y(t - t_0) = x(t - t_0)^2 \)

BIBO stable: yes, since if \( |x(t)| \leq B_1 < \infty \) then \( |y(t)| \leq B_1^2 \leq \infty \).

Invertible: no, since any negative input value is lost, and we can not recover the original sign of the input.

Memoryless: yes, since \( y(t) \) depends on no other value of \( x(t) \) other than at time \( t \).

Causal: yes, since all memoryless systems are causal.

(1b) (5pts) Consider the system described by the x-y-relation as follows:

\[ y(t) = t^2 x(t) \]  

\[ (0.2) \]

Is this system: i) linear, ii) time-invariant, iii) BIBO stable, iv) invertible, v) memoryless, vi) causal? **Circle yes/no/unknown for each and include brief justification.**

Linear: yes, since with input \( a_1 x_1(t) + a_2 x_2(t) \) produces output \( t^2(a_1 x_1(t) + a_2 x_2(t)) = a_1 t^2 y_1(t) + a_2 t^2 y_2(t) \) using the usual definitions of \( y_1(t) \) and \( y_2(t) \).

Time-invariant: no, since the system depends on values of \( x(t) \) at specific values of time via the \( t^2 \) factor. I.e., shifting the input produces output \( t^2 x(t - t_0) \) but shifting the output produces \( t - t_0 \) grows in \( t \) without bound.

BIBO stable: no, since the output is unbounded for many bounded inputs (e.g., consider \( x(t) = u(t) \), then \( y(t) = t^2 u(t) \) grows in \( t \) without bound).

Invertible: No, since the value at \( t = 0 \) is lost. I.e., \( y(0) = 0 \) regardless of \( x(0) \), so we can’t get back \( x(0) \).

Memoryless: yes, since \( y(t) \) depends values of \( x(t) \) only at time \( t \).

Causal: yes, all memoryless systems are causal.

(1c) (5pts) Consider the system described by the x-y-relation as follows:

\[ y(t) = e^{j \omega_0 t} x(t) - j x(t - t_0) \]  

\[ (0.3) \]

where \( t_0 \) and \( \omega_0 \) are constants.

Is this system: i) linear, ii) time-invariant, iii) BIBO stable, iv) memoryless, v) causal? **Circle yes/no/unknown for each and include brief justification;**
Linear: yes, since \( S[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 y_1(t) + \alpha_2 y_2(t) \). The reason this is true is that

\[
S[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = e^{j\omega_0 t} \left( \alpha_1 x_1(t) + \alpha_2 x_2(t) \right) - j \left( \alpha_1 x_1(t - t_0) + \alpha_2 x_2(t - t_0) \right) \\
= \alpha_1 \left( e^{j\omega_0 t} x_1(t) - j x_1(t - t_0) \right) + \alpha_2 \left( e^{j\omega_0 t} x_2(t) - j x_2(t - t_0) \right) \\
= \alpha_1 y_1(t) + \alpha_2 y_2(t)
\] (0.4)

Time-invariant: No, since when we shift the output \( y(t - t_0) \) we’ll shift the \( e^{j\omega_0 t} \) factor in the first term, but this won’t happen when we shift the input.

BIBO stable: yes, since nothing is going to make the output infinite as long as \( x(t) \) is finite.

Memoryless: No, since \( y(t) \) depends on values of the input at times other than time \( t \).

Causal: yes, since \( y(t) \) depends on values of \( x(t) \) only up to time \( t \).

(1d) (5pts) Consider the linear system described by the input-output relations (i.e., input \( x_i(t) \) produces output \( y_i(t) \), for \( i \in \{1, 2\} \)) in the following picture:

![Input-Output Relations](image)

Is this system: i) time-invariant ii) BIBO stable, iii) invertible, iv) memoryless, v) causal? Circle yes/no/unknown for each and include brief justification;

We are given that the system is linear.
Time-invariant: no, as since \( x_2(t) \) is a shifted scaled version of \( x_1(t) \) but the outputs are not shifted and scaled versions of each other.
BIBO stable: unknown, since we don’t know if an an bounded input could produce an unbounded output. Consider some bounded input not shown, it could still produce an unbounded output.
Invertible: again unknown, we don’t know enough about the system.
Memoryless: It could be, but again not enough is known to determine if for certain the system is memoryless. We can come up with two systems one memoryless and the other non-memoryless both of which correspond to the system.
Causal: Not causal, for the same reason.

Problem # 2 (20 points)

For full credit in this problem, you must be extremely precise, but you must also give short and extremely clear answers. We know from class that any periodic signal \( x(t) \) with period \( T = 2\pi/\omega_0 \) can be represented using the Fourier series of the form:

\[
x(t) = \sum_{k} C_k e^{j\omega_0 t}
\]

You have an old friend, Bob, who took EE235 at a different university and he claims that you can use the Fourier series, and its standard analysis/synthesis equations, to perfectly represent non-periodic signals. In this problem, we are going to argue why Bob is not correct.
(2a) (5pts)

Using the Fourier series analysis equation, explain to Bob where things will go wrong if we try to analyze a signal \( x(t) \) that is now not periodic. Be extremely precise in your answer — explain precisely where and why this analysis equation does not work in this case. Be clear and brief.

The key issue that you need to get here is that there is no period, so if we look at the analysis equation

\[
C_k = \int_T x(t)e^{-j\omega_0 t}dt
\]

which says to integrate over one period, but since \( x(t) \) is not periodic, there is no period to integrate over. If we choose a particular range \([0, T]\) or \([-T/2, T/2]\), then we’ll potentially get different versions of \( C_k \) for each range we choose, so there would not be a unique set of \( C_k \).

(2b) (5pts)

Using the Fourier series synthesis equation, come up with another brief explanation to Bob where representing a non-periodic \( x(t) \) using the Fourier series will go wrong. Consider the synthesis equation as we have described it in class, i.e., as a countably infinite sum of weighted complex exponentials. What property must such sums possess (and argue why they possess this property) which leads to the reason that they do not work. Be clear and brief.

Considering the synthesis equation, we see that we are summing weighted complex exponentials which are all periodic with some fundamental period \( T \). Summing such signals will always yield a periodic signal with period \( T/m \) for some positive integer \( m \), therefore a non-periodic signal cannot be represented in this equation.

After Bob finally sees the light, he claims that you can just truncate the non-periodic \( x(t) \) to the time range \([0, T]\). I.e., Bob claims that the Fourier series can be applied to the signal \( \bar{x}(t) = x(t) \times [u(t) - u(t - T)] \).

(2c) (5pts)

Is Bob’s new \( \bar{x}(t) \) periodic for all signals \( x(t) \)? Briefly and clearly show all work giving why or why not.

Bob’s new \( \bar{x}(t) \) is not periodic. We can see this since it is a rectangle of length \( T \) (i.e., \( u(t) - u(t - T) \)) multiplied by \( x(t) \), so all values outside the range \([0, T]\) will be zero.

(2d) (5pts)

Consider the coefficients:

\[
\bar{C}_k = \frac{1}{T} \int_0^T \bar{x}(t)e^{-j\omega_0 t}dt \quad (0.7)
\]

If you answered yes to question (2c), compute the Fourier series coefficients \( \bar{C}_k \) for \( \bar{x}(t) \) as shown above, assuming that \( x(t) = e^{j\omega_0 t} + j\pi/2e^{j\pi t} \). If, on the other hand, you answered no to question (2c), then come up with an expression for a time signal \( \hat{x}(t) \) that has Fourier series coefficients \( \bar{C}_k \) as shown in the above equation (i.e., in this case, start your answer with \( \hat{x}(t) = \ldots \)).

The Fourier series coefficients computed in this equation integrate over the time range \([0, T]\), so this equation would not know if the signal is periodic and we happen to integrate over one period, or if the signal is not periodic and we are only integrating over the first \( T \) time units. In either case, we can come up with a periodic signal with period \( T \) that has the same values in \([0, T]\) as does \( x(t) \) as follows,

\[
\hat{x}(t) = \sum_k \bar{x}(t - kT)
\]

Problem # 3 (20 points)

Given an LTI system \( S[\cdot] \), we defined \( h(t) \) (the impulse response) as the response of the system to input \( \delta(t) \). We also defined \( s(t) \) (the step response) as the response of the system to input \( u(t) \), and since \( u(t) = \int_{-\infty}^t \delta(t)dt \), we can see
that $s(t) = \int_{-\infty}^{t} h(\tau)d\tau$. Consider now input $v(t) = \int_{-\infty}^{t} u(\tau)d\tau$ which is a ramp, as shown in the following figure.

We then compute $S[v(t)] = r(t)$ which we shall in this problem call the ramp response of the LTI system $S$.

(3a) (7 pts)

Find the ramp response for an LTI system with impulse response $h(t) = u(t) - u(t - t_0)$ for $t_0 > 0$. To do this, briefly and clearly identify the time regions in terms of constraints on $t$ corresponding to each of the different portions of the solutions you will get. Also, plot the final $r(t)$ in as simple as possible notation.

Our goal is to compute $v(t) * h(t)$. We first plot $v(\tau)$ and $h(t - \tau)$ and see that we have three time regions, $t < 0$, $0 < t < t_0$, and also $t > t_0$. We get:

$$r(t) = v(t) * h(t) = \int_{-\infty}^{\infty} h(t - \tau)v(\tau)d\tau$$

$$= \begin{cases} 
0 & t < 0 \\
\frac{1}{2}t^2 & 0 < t < t_0 \\
tt_0 - \frac{1}{2}t_0^2 & t > t_0
\end{cases}$$

(3b) (7 pts)

Is it possible to process $r(t)$ to recover $h(t)$? If so, clearly and briefly provide those operations and briefly verify that your solution is correct? If not, explain why not?

Due to the construction of $v(t)$ in terms of double integrals of the impulse and the fact that the system is LTI, we can see that $r(t)$ is also the double integral of $h(t)$. Therefore, we have that $h(t) = \frac{d^2}{dt^2}r(t)$. To verify, we get:

$$\frac{d}{dt} r(t) = \begin{cases} 
0 & t < 0 \\
t & 0 < t < t_0 \\
t_0 & t > t_0
\end{cases}$$

and

$$\frac{d^2}{dt^2} r(t) = \begin{cases} 
0 & t < 0 \\
1 & 0 < t < t_0 \\
0 & t > t_0
\end{cases}$$

which we see to be $h(t)$.

(3c) (6 pts)

We know that the impulse response $h(t)$ of an LTI system can be used to infer other system properties (such as memoryless, causality, BIBO stability, etc). Starting from just the ramp-response $r(t)$ of an LTI system, is it possible to infer the following system properties? Clearly and briefly argue why or why not.
Since we can go from \( r(t) \) back to \( h(t) \) as shown in the previous part of this question, and since we can get all information about the system from \( h(t) \), this means that \( r(t) \) also has all information, so we can find out if the system is memoryless, causal, and BIBO stable.

Regarding BIBO stability, note that this property of a system says that for a system to be BIBO stable, it must produce a bounded output for a bounded input. It does not say anything about what happens if the input is unbounded, as is \( v(t) \) above.

**Problem #4 (20 points)**

The following (inefficient) MATLAB function takes two vectors, \( a \) and \( b \) and creates a scaled sum of the two. The scaled sum is in the range of \([-1.5, 1.5]\). The following is known about vectors \( a \) and \( b \): 1) \( a \) and \( b \) have the same dimensions (they are either both row or both column vectors), and 2) they contain only finite numbers.

```matlab
function [sum, scale, scaled] = scaledsum(a,b);
% This function takes in a and b, two arbitrary real vectors with the
% same dimensions. It first calculates the unscaled sum of the two
% vectors and stores this in the variable sum. Then it finds the entry
% with the maximum magnitude, and stores this in the variable scale.
% The code then scales the sum to be in the range of [-1.5, 1.5], and
% this is stored in the variable scaled.
for i=1:length(a)
    sum(i) = a(i) + b(i);
end
minimum=0;
maximum=0;
for i=1:length(sum)
    if sum(i) > maximum
        maximum = sum(i);
    elseif sum(i) < minimum
        minimum = sum(i);
    end
end
scale = max([maximum -minimum]);
for i=1:length(sum)
    scaled(i) = sum(i)/scale* 1.5;
end
```

Your task is to re-write this matlab code so that it is much more concise and efficient. In particular, write some equivalent matlab code that: 1) Assigns the proper values to the \( \text{sum} \), \( \text{scale} \), and \( \text{scaled} \) variables; 2) uses no for loops; and 3) uses a minimal number of lines. Note that a correct answer can do this in only 3 lines of matlab code, where each line has at most 26 characters. For full credit, briefly justify why your code is correct.

The following is one possible answer.

```matlab
sum = a + b;
scale = max ( abs(sum) );
scaled = sum / scale * 1.5;
```

**Problem #5 (20 points)**

In this problem, all signals and constants are, as usual, complex valued. Recall the definition of the linearity property of a system \( S[\cdot] \), which states that the system must satisfy the scaling property \( S[\alpha x(t)] = \alpha S[x(t)] \) and also the additivity property \( S[x_1(t) + x_2(t)] = S[x_1(t)] + S[x_2(t)] \). If we combine these two properties, we get the general superposition property of linear systems. Superposition states, in its general form, that given general complex input signals \( x_1(t) \) and \( x_2(t) \) to \( S[\cdot] \) with corresponding complex output signals \( y_1(t) \) and \( y_2(t) \), then:

\[
\alpha_1 y_1(t) + \alpha_2 y_2(t) = S[\alpha_1 x_1(t) + \alpha_2 x_2(t)]
\]
where $\alpha_1$ and $\alpha_2$ are arbitrary complex constants. Clearly, if both the scaling and additivity property hold, then superposition holds.

In this problem, we investigate if the scaling and additivity property are redundant or not. In other words, given a system that satisfies additivity, either prove or disprove that the system must also satisfy scaling.

Specifically, suppose the system $S[\cdot]$ satisfies the additivity property. Must the system then satisfy scaling? If so, show it to be true, and if not, come up with a very specific, counter example system that satisfies additivity but does not satisfy scaling.

A system that satisfies additivity need not satisfy scaling. To see this, consider the system that takes only the real part of its input, i.e. $S[x(t)] = \Re[x(t)]$. Then for complex valued signals $x_1(t) = x_{1R}(t) + jx_{1I}(t)$ and $x_2(t) = x_{2R}(t) + jx_{2I}(t)$ we get

\begin{align*}
y(t) &= S[x_1(t) + x_2(t)] \quad (0.8) \\
    &= S[x_{1R}(t) + jx_{1I}(t) + x_{2R}(t) + jx_{2I}(t)] \quad (0.9) \\
    &= x_{1R}(t) + x_{2R}(t) \quad (0.10) \\
    &= S[x_1(t)] + S[x_2(t)] \quad (0.11) \\
    &= S[x_1(t)] + S[x_2(t)] \quad (0.12)
\end{align*}

On the other hand, if $\alpha$ is a complex number, then $S[\alpha x(t)]$ will always be real, but $\alpha S[x(t)]$ might be complex, so scaling might not hold.